

The Adomian Polynomials and the New Modified Decomposition Method for BVPs of nonlinear ODEs

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Abstract

In this paper we consider the new algorithm for the Adomian polynomials and the new modified decomposition method for solving boundary value problems of nonlinear ordinary differential equations. In the new method, the recursion scheme does not involve undetermined coefficients. Thus we avoid the complications resulting from the necessity of evaluating such undetermined coefficients at each stage of approximation. Furthermore, the recursion scheme can embed a convergence parameter to efficiently calculate the sequence of the analytical approximate solutions.

Keywords: *Adomian Decomposition Method; Adomian Polynomials; Boundary Value Problem; Ordinary Differential Equation*

1 INTRODUCTION

The Adomian decomposition method (ADM) ^[1-6] is a powerful tool for solving linear or nonlinear functional equations. The method give analytic approximations by a recursive manner. Applying the ADM to the boundary value problems (BVPs) for ordinary differential equations (ODEs) can avoid using the Green function concept, which greatly facilitates analytic approximations and numerical computations.

There are several different resolution techniques based on the ADM for solving BVPs for nonlinear ODEs, such as the double decomposition method ^[4, 7, 8] and the Duan-Rach modified decomposition method ^[9]. The double decomposition method decomposes the solutions, the nonlinearities and the undetermined coefficients into series before designing the recursion scheme for the solution components. The Duan-Rach modified decomposition method excludes all undetermined coefficients when computing successive solution components. We note that parametrized recursion scheme can be used to achieve simple-to-integrate series, fast rate of convergence and extended region of convergence ^[9-11].

We remark that the convergence of the Adomian series has already been proven by several investigators ^[5, 12, 13]. For example, Abdelrazec and Pelinovsky ^[13] have published a rigorous proof of convergence for the ADM under the aegis of the Cauchy-Kovalevskaya theorem for initial value problems. A key concept is that the Adomian decomposition series is a computationally advantageous rearrangement of the Banach-space analog of the Taylor expansion series about the initial solution component function, which permits solution by recursion. Furthermore convergence of the ADM is not limited to cases when only the fixed-point theorem applies, which is far too restrictive for most physical applications. Different classes and generalization of the Adomian polynomials were presented in ^[3, 5, 14-16]. New applications and numerical methods based on the ADM were developed in ^[17-20].

2 ADOMIAN POLYNOMIALS

The decomposition method decomposes the solution $u(x)$ and the nonlinearity Nu into series

$$u(x) = \sum_{n=0}^{\infty} u_n(x), Nu = \sum_{n=0}^{\infty} A_n, \quad (1)$$

where $A_n = A_n(u_0(x), u_1(x), \dots, u_n(x))$ are the Adomian polynomials

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} N\left(\sum_{k=0}^{\infty} u_k \lambda^k\right) \Big|_{\lambda=0}, \quad n \geq 0. \quad (2)$$

We list the first five Adomian polynomials for the nonlinearity $Nu = f(u)$,

$$\begin{aligned} A_0 &= f(u_0), A_1 = f'(u_0)u_1, A_2 = f'(u_0)u_2 + f''(u_0)\frac{u_1^2}{2!}, \\ A_3 &= f'(u_0)u_3 + f''(u_0)u_1u_2 + f'''(u_0)u_1^3\frac{3!}{3!}, \\ A_4 &= f'(u_0)u_4 + f''(u_0)\left(\frac{u_2^2}{2!} + u_1u_3\right) + f'''(u_0)\frac{u_1^2u_2}{2!} + f^{(4)}(u_0)\frac{u_1^4}{4!}. \end{aligned}$$

Several algorithms [1, 5, 6, 21-27] for symbolic programming have since been devised to generate the Adomian polynomials quickly and to high orders. New, more efficient algorithms and subroutines in MATHEMATICA for rapid computer-generation of the Adomian polynomials are provided in [10, 15, 28-30]. Here we list the Corollary 3 algorithm in [30] as follows.

$$A_0 = f(u_0), \quad A_n = \sum_{k=1}^n C_n^k f^{(k)}(u_0), \quad n \geq 1, \quad (3)$$

where the coefficients C_n^k are defined recursively as

$$\begin{aligned} C_n^1 &= u_n, \quad n \geq 1, \\ C_n^k &= \frac{1}{n} \sum_{j=0}^{n-k} (j+1) u_{j+1} C_{n-1-j}^{k-1}, \quad 2 \leq k \leq n. \end{aligned} \quad (4)$$

We remark that this algorithm does not involve the differentiation operator for the coefficients C_n^k , but only requires the elementary operations of addition and multiplication, and is thus eminently convenient for computer algebra systems such as MATHEMATICA, MAPLE or MATLAB.

3 THE NEW MODIFIED DECOMPOSITION METHOD

We display the new modified decomposition method by considering the BVP for the second-order nonlinear ODE,

$$Lu = Nu + g(x), \quad a < x < b, \quad (5)$$

$$u(a) = \alpha, \quad u(b) = \beta, \quad (6)$$

where $L = \frac{d^2}{dx^2}$, $g(x)$ is a prescribed continuous function and Nu is an analytic nonlinearity. We assume the solution of the BVP exists uniquely.

For the BVP in Eqs. (5) and (6), we take the inverse linear operator as $L^{-1}(\cdot) = \int_a^x \int_a^x (\cdot) dx dx$. Applying the operator L^{-1} to both sides of Eq. (5) yields

$$u(x) - u(a) - u'(a)(x-a) = L^{-1}Nu + L^{-1}g(x). \quad (7)$$

Let $x = b$ in Eq. (7) and then solve for $u'(a)$, then

$$u'(a) = \frac{u(b) - u(a) - [L^{-1}Nu]_{x=b} - [L^{-1}g(x)]_{x=b}}{b-a}, \quad (8)$$

where $[L^{-1}(\cdot)]_{x=b} = \int_a^b \int_a^x (\cdot) dx dx$. Substituting Eq. (8) into Eq. (7) yields

$$u(x) = u(a) + \frac{u(b) - u(a)}{b-a}(x-a) - \frac{x-a}{b-a}[L^{-1}g]_{x=b} + L^{-1}g + L^{-1}Nu - \frac{x-a}{b-a}[L^{-1}Nu]_{x=b}. \quad (9)$$

Thus the right hand side of Eq. (9) does not contain the undetermined coefficient $u'(a)$. Next, we decompose the solution and the nonlinearity

$$u(x) = \sum_{n=0}^{\infty} u_n(x), Nu = \sum_{n=0}^{\infty} A_n, \quad (10)$$

From Eq. (9), the solution components are determined by the modified recursion scheme

$$u_0 = u(a) + \frac{u(b) - u(a)}{b - a}(x - a) - \frac{x - a}{b - a} [L^{-1}g]_{x=b} + L^{-1}g, \quad (11)$$

$$u_{n+1} = L^{-1}A_n - \frac{x - a}{b - a} [L^{-1}A_n]_{x=b}, n \geq 0. \quad (12)$$

The n -term approximate solution is denoted as $\phi_n(x) = \sum_{k=0}^{n-1} u_k$. We can also design other recursion schemes including the parametrized recursion scheme [9-11].

Example 1. Consider the nonlinear BVP

$$u''(x) = e^u, 0 \leq x \leq 1, \quad (13)$$

$$u(0) = 0, u(1) = 0. \quad (14)$$

The exact solution is [31]

$$u^*(x) = 2 \ln(k \sec \frac{k(2x-1)}{4}) - \ln(2), \quad (15)$$

where k satisfies $k \sec \frac{k}{4} = \sqrt{2}$, to 6 significant figures, $k = 1.33606$.

We decompose the solution $u(x)$ and the nonlinearity e^u as

$$u(x) = \sum_{n=0}^{\infty} u_n(x), e^u = \sum_{n=0}^{\infty} A_n,$$

where the Adomian polynomials for $f(u) = e^u$ are

$$A_0 = e^{u_0}, A_1 = e^{u_0} u_1, A_2 = e^{u_0} (u_2 + \frac{u_1^2}{2}), A_3 = e^{u_0} \left(\frac{u_1^3}{6} + u_1 u_2 + u_3 \right),$$

$$A_4 = e^{u_0} \left(\frac{u_1^4}{24} + \frac{1}{2} u_1^2 u_2 + \frac{u_2^2}{2} + u_1 u_3 + u_4 \right), \dots$$

According to the above procedure, the solution components are determined by the modified recursion scheme

$$u_0 = 0, u_n = L^{-1}A_{n-1} - x[L^{-1}A_{n-1}]_{x=1}, n = 1, 2, \dots$$

By computation we have

$$u_1 = -\frac{x}{2} + \frac{x^2}{2}, u_2 = \frac{x}{24} - \frac{x^3}{12} + \frac{x^4}{24}, u_3 = -\frac{x}{160} + \frac{x^3}{144} + \frac{x^4}{96} - \frac{x^5}{60} + \frac{x^6}{180}, \dots$$

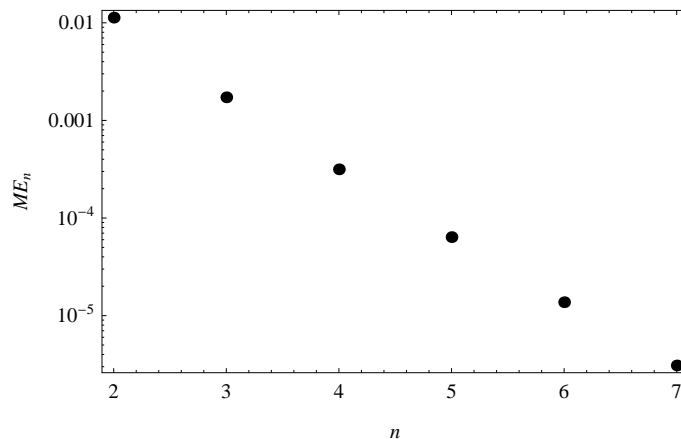


FIG. 1: LOGARITHMIC PLOTS OF MAXIMAL ERRORS ME_n VERSUS n ($n = 2, 3, 4, 5, 6, 7$).

To examine the convergence of the n -term approximation $\phi_n(x) = \sum_{k=0}^{n-1} u_k$, we consider the maximal errors

$$ME_n = \max_{0 \leq x \leq 1} |\phi_n(x) - u^*(x)|, \quad (16)$$

which are computed by using MATHEMATICA. In Fig. 1 we display the logarithmic plots of the maximal errors ME_n versus n for $n = 2$ to 7. The data points lie almost on a straight line, which means that the maximal errors decrease approximately at an exponential rate.

Example 2. Consider the nonlinear BVP

$$u'' = 6u^2, \quad 0 \leq x \leq 1, \quad (17)$$

$$u(0) = 1, \quad u(1) = 1/4. \quad (18)$$

The exact solution is $u^*(x) = (1+x)^{-2}$.

According to our procedure, we have

$$u = 1 - \frac{3}{4}x + 6L^{-1}u^2 - 6x[L^{-1}u^2]_{x=1}, \quad (19)$$

where $L^{-1}(\cdot) = \int_0^x \int_0^x (\cdot) dx dx$. The Adomian polynomials for the nonlinearity $f(u) = u^2$ with the decomposition $u = \sum_{n=0}^{\infty} u_n$ are

$$A_0 = u_0^2, \quad A_1 = 2u_0u_1, \quad A_2 = 2u_0u_2 + u_1^2, \quad \dots, \quad A_n = \sum_{k=0}^n u_k u_{n-k}.$$

We have checked that the recursion scheme

$$u_0 = 1 - \frac{3}{4}x,$$

$$u_{n+1} = 6L^{-1}A_n - 6x[L^{-1}A_n]_{x=1}, \quad n \geq 0,$$

yields a divergent series. But if we use the parametrized recursion scheme

$$u_0 = c + 1 - \frac{3}{4}x,$$

$$u_{n+1} = -\frac{c}{2^{n+1}} + 6L^{-1}A_n - 6x[L^{-1}A_n]_{x=1}, \quad n \geq 0,$$

convergent approximate solutions can be obtained for appropriate values of c . For example, we take $c = -0.4$ and calculate the solution components u_n . The maximal errors $ME_n = \max_{0 \leq x \leq 1} |\phi_n(x) - u^*(x)|$, for $n = 1$ through 16, are listed in Table 1, where the sequence $\{ME_n\}$ decreases monotonically.

TABLE 1: FOR $c = -0.4$, MAXIMAL ERRORS ME_n FOR $n = 1$ THROUGH 16.

n	1	2	3	4	5	6	7	8
ME_n	0.4	0.2	0.1	0.05	0.025	0.0125	0.00625	0.003125
n	9	10	11	12	13	14	15	16
ME_n	0.001563	0.000781	0.000391	0.000195	0.000098	0.000049	0.000024	0.000012

4 CONCLUSIONS

We have presented the new modified decomposition method for solving BVPs of nonlinear ODEs. The recursion scheme can embed a convergence parameter to efficiently calculate the sequence of the analytical approximate solutions. In the new method, the undetermined coefficients are inserted due to the nature of the boundary conditions. Thus we avoid the complications resulting from the necessity of evaluating such undetermined coefficients at each stage of approximation. The overall efficiency of our new modification of the ADM is further enhanced by the new

algorithms and subroutines [10, 28-30] for generating the Adomian polynomials quickly and to high orders.

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