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Oscillation Criteria for Even Order Functional Differential Equations with Damped Term

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Abstract

This paper investigates a class of even order functional differential equations with damped term, and derives two new oscillatory criteria of solution.

Keywords: Functional Differential Equation; Even Order; Damped; Oscillation

1 INTRODUCTION

Oscillation of even order functional differential equations has been studied extensively. It is caused by the applications such as physics, chemistry, phenomena arising in engineering, economy, and science; for example, [1–6]. Recently, more and more authors have paid their attentions. We refer to the monographs [1, 2]. However, most of the literatures dealing with equations do not contain the damped term. It seems that very little is known about certain equations with damped term, (for example, the literatures [3-5]), especially the case of containing distributed deviating arguments. In this paper, we deal with a class of even order functional differential equations with damped of the form

$$(r(t)\left[x(t)+\int_c^d c(t,\eta)x\left[h(t,\eta)\right]d\mu(\eta)\right]^{(n-1)})' + p(t)\left[x(t)+\int_c^d c(t,\eta)x\left[h(t,\eta)\right]d\mu(\eta)\right]^{(n-1)}$$
$$+\int_a^d q(t,\xi)f(x(t),\xi,x\left[g(t,\xi)\right])d\sigma(\xi)=0, t\ge t_0>0 \tag{1}$$

and establish two new oscillatory criteria of solution, where *n* is an even number, and the following conditions(*H*) are always assumed to hold:

$(H1)$ $(pt)\in C([t_0,\infty),R_+), r(t)\in C([t_0,\infty),R_+), r'(t)>0, q(t,\xi)\in C([t_0,\infty),R_+)$,

$C(t,\eta)\in C([t_0,\infty)\times[a,b],R_+)$ is not identically zero on any $[t_\mu,\infty)\times[a,b], t_\mu\ge t_0$;

$(H2)$ $g(t,\xi)\in C([t_0,\infty)\times[a,b],R), g(t,\xi)<t, \xi\in[a,b]$,and $\liminf_{t\to\infty,\xi\in[a,b]}, g(t,\xi)=\infty$;

$(H3)$ $f(u_1,u_2,u_3)\in C(R\times R\times R,R)$ have same sign with u_1,u_2,u_3, when u_1,u_2,u_3 have same sign;

$(H4)$ $\sigma(\xi)\in([a,b],R)$ is nondecreasing, the integral of equation (1) is a Stieltjes one.

We restrict out attention to proper solutions of equation (1), i.e to nonconstant solutions existing on $[t,\infty)$ for some $T\ge t_0$ and satisfying $\sup_{t\ge T}|x(t)|>0$. A proper solution $x(t)$ of equation (1) is called oscillatory if it does not the largest zero, otherwise, it is called nonoscillatory.

The following three Lemmas will be needed in the proof of our results:

Lemma 1. [6] Let $u(t)$ be a positive and *n* times differentiable function on $[0,\infty)$. If $u^{(n)}(t)$ is a constant sign and not identically zero on any ray $[t_1,\infty), t_1>0$, then there exists a $t_\mu\ge t_1$ and an integer $l(0\le l\le n)$, with $n+l$ even for $u(t)u^{(n)}(t)\ge 0$ or $n+l$ odd for $u(t)u^{(n)}(t)\le 0$; and for $t\ge t_\mu$,

$$u(t)u^{(k)}(t)>0, 0\le k\le l;\quad (-1)^{(k-l)}u(t)u^{(k)}(t)>0, l\le k\le n \tag{2}$$

Lemma 2. [7] Suppose that the conditions of Lemma l are satisfied, and then for any constant $\theta\in(0,1)$ and sufficiently large *t*, there exists a constant *M* satisfying

$$\left|\mu'(\theta(t)\right| \geq Mt^{(n-2)}\left|\mu^{(n-1)}(t)\right| \tag{3}$$

Similar to the proof of literature [8], we have

Lemma 3. Suppose that $x(t)$ is a nonoscillatory solution of equation (1). If

$$\lim_{t\to\infty}\int_{t_1}^{t}\exp(-\int_{t_1}^{s}p(\tau)d\tau)ds = +\infty, t \geq t_0 \tag{4}$$

Then $$x(t)x^{(n-1)}(t) > 0 \ , \ for \ any \ large \ t \tag{5}$$

2 MAIN RESULTS

Theorem 1. Suppose that

$(H5)$ There exists a function $\sigma(t) \in C'([t-0,\infty),(0,\infty)$ *satisfying* $\sigma(t) \leq g(t,\xi) \leq t$, $0 < c \leq \sigma'(t) \leq 1$, and $\lim_{t\to\infty}\sigma(t) = \infty$;

$(H6)$ $\frac{\partial f}{\partial u_i}$ exists, and $\frac{\partial f}{\partial u_i} \geq c_i > 0$, where $c_i > 0$ are some constants, $i = 1,1,3$. If for any large T

$$\limsup_{t\to\infty}\frac{1}{t^{m-1}}\int_T^t (t-u)^{m-3}u^k[\int_a^b (t-u)^2 q(u,\xi)d\sigma(\xi)$$

$$-\frac{[(t-u)\frac{pr(u)}{\gamma(u)} - \frac{k}{u} + m - 1]^2}{4M_1\frac{1}{\gamma(u)}\sigma^{(n-2)}(u)}]du = \infty \tag{6}$$

in which M_1 is a constant, then every solution of equation (1) is oscillatory.

Proof. We assume that equation (1) has a nonoscillatory solution $x(t)$. Without loss of generality, we may suppose that $x(t) > 0$ for all large *t*. The case of $x(t) < 0$ can be considered by the same method. From (*H1*), (*H2*), and (*H3*), there exists a $t_1 \geq t_0$ such that $x[h(t,\eta)] > 0$, $f(x(t),\xi,x[g(t,\xi)] > 0$ for $t \geq t_1$ and $\xi \in [a,b]$.

Let

$$z(t) = x(t) + \int_c^d c(t,\eta)x[h(t,\eta)]d\mu(\eta) \tag{7}$$

Thus equation (1) turns into

$$(r(t)\mathrm{z}^{(n-1)}(t))' + p(t)(\mathrm{z}^{(n-1)}(t)) + \int_a^b q(t,\xi)f(x(t),\xi,x[g(t,\xi)])d\sigma(\xi) = 0, t \geq t_0 > 0$$

Then, From *(H1)*, we obtain $\mathrm{z}(t) \geq 0, \mathrm{z}(t) \geq x(t)$. Using $\mathrm{z}(t) \geq x(t)$, and from *(H6)*, we can get $f(x(t),\xi,\mathrm{z}[g(t,\xi)]) \geq f(x(t),\xi,x[g(t,\xi)])'$ Thus

$$(r(t)\mathrm{z}^{(n-1)}(t))' + p(t)(\mathrm{z}^{(n-1)}(t)) + \int_a^b q(t,\xi)f(x(t),\xi,x[g(t,\xi)])d\sigma(\xi) \leq 0, t \geq t_0 > 0 \tag{8}$$

From the assumption of (*H1*), we have $\mathrm{z}[t]) \geq x(t) > 0, (r(t)\mathrm{z}^{(n-1)}(t))' \leq 0$, for $t \geq t_1$, and $(r(t)\mathrm{z}^{(n-1)}(t))'$ is not eventually zero, thus, we have

$$(r(t)\mathrm{z}^{(n-1)}(t))' = r'(t)\mathrm{z}^{(n-1)}(t) + r(t)\mathrm{z}^{(n)}(t) \leq 0 \tag{9}$$

Similar to the proof of Lemma 3, we have $\mathrm{z}^{(n-1)}(t) > 0$. Thus from (9), we obtain $\mathrm{z}^{(n)}(t) \leq 0$ for $t \geq t_1$.

From Lemma 1, there exists a $t_2 \geq t_1$ and an odd number $(0 < l < n)$, such that for $t \geq t_2$

$$\mathrm{z}^{(k)}(t) > 0, 0 \leq k \leq l; (-1)^{(k-1)}\mathrm{z}^{(k)}(t) > 0, l \leq k \leq n$$

By choosing *k* = 1, we have

$$\mathrm{z}'(t) > 0, t \geq t_2 \tag{10}$$

It follows that Lemma 2, there exists a constant $M>0$ and $t_3 \ge t_2$ such that

$$z'(\frac{t}{2}) \ge Mt^{(n-2)} z^{(n-1)}(t), z'(\frac{\sigma(t)}{2}) \ge M\sigma^{(n-2)}(t) z^{(n-1)}(t) \tag{11}$$

From $(H6), z(t) \ge x(t)$, we have $f(z(t),\xi,z[g(t,\xi)] \ge f(x(t),\xi,x[g(t,\xi)])$, thus

$$(r(t)z^{(n-1)}(t))'+p(t)(z^{(n-1)}(t))+\int_a^b q(t,\xi)f(z(t),\xi,z[g(t,\xi)])d\sigma(\xi) \le 0, t \ge t_0 > 0 \tag{12}$$

Let

$$y(t)=\frac{t^k z^{(n-1)}(t)}{f(z\left[\frac{t}{2}\right],\frac{a}{2},z\left[\frac{\sigma(t)}{2}\right])} \tag{13}$$

Then we conclude that

$$y'(t)=\frac{t^k(r(t)z^{(n-1)}(t))'}{f(z\left[\frac{t}{2}\right],\frac{a}{2},z\left[\frac{\sigma(t)}{2}\right])}+\frac{kt^{(k-1)}r(t)z^{(n-1)}(t)}{f(z\left[\frac{t}{2}\right],\frac{a}{2},z\left[\frac{\sigma(t)}{2}\right])}-\frac{1}{2}\frac{t^k r(t)z^{(n-1)}(t)}{f^2(z\left[\frac{t}{2}\right],\frac{a}{2},z\left[\frac{\sigma(t)}{2}\right])}$$

$$\times(\frac{\partial f(z\left[\frac{t}{2}\right],\frac{a}{2},z\left[\frac{\sigma(t)}{2}\right])}{\partial z\left[\frac{t}{2}\right]}+\frac{kt^{(k-1)}r(t)z^{(n-1)}(t)}{f(z\left[\frac{t}{2}\right],\frac{a}{2},z\left[\frac{\sigma(t)}{2}\right])}-\frac{1}{2}\frac{t^k r(t)z^{(n-1)}(t)}{f^2(z\left[\frac{t}{2}\right],\frac{a}{2},z\left[\frac{\sigma(t)}{2}\right])}$$

In view of (*H3*) and (*H4*), and noting that

$$\frac{-\int_a^b q(t,\xi)f(z(t),\xi,z[g(t,\xi)])d\sigma(\xi)}{f(z\left[\frac{t}{2}\right],\frac{a}{2},z\left[\frac{\sigma(t)}{2}\right])} \le \frac{-\int_a^b q(t,\xi)f(z(t),a,z(\sigma(t))d\sigma(\xi)}{f(z\left[\frac{t}{2}\right],\frac{a}{2},z\left[\frac{\sigma(t)}{2}\right])}$$

$$\le -\int_a^b q(t,\xi)d\sigma(\xi) \tag{14}$$

We conclude that

$$y'(t) \le -t^k\int_a^b q(t,\xi)d\sigma(\xi)-(\frac{p(t)}{r(t)}-\frac{k}{t})y(t)$$

$$-\frac{1}{2}(c_1Mt^{(n-2)}+c_3cM\sigma^{(n-2)}(t))z^{(n-1)}(t)\times\frac{t^k r(t)z^{(n-1)}(t)}{f^2(z\left[\frac{t}{2}\right],\frac{a}{2},z\left[\frac{\sigma(t)}{2}\right])}$$

$$\le -t^k\int_a^b q(t,\xi)d\sigma(\xi)-(\frac{p(t)}{r(t)}-\frac{k}{t})y(t)-M_1\sigma^{(n-2)}(t)y^2(t)\frac{t^{-k}}{r(t)} \tag{15}$$

in which $M_1=\frac{1}{2}M(c_1+c_3c)$. From (14), for any large $t>t_3$, we conclude that

$$\int_{t_3}^t (t-u)^{(m-1)}y'(u)du \le -\int_{t_3}^t\int_a^b (t-u)^{(m-1)}u^k q(u,\xi)d\sigma(\xi)du \le -(\xi)-(\xi)$$

$$-\int_{t_3}^t (t-u)^{(m-1)}-(\frac{p(u)}{r(u)}-\frac{k}{u})y(u)du$$

$$-\int_{t_3}^t M_1(t-u)^{(m-1)}\sigma^{(n-2)}(u)\frac{u^{-k}}{r(u)}y^2(u)du$$

By part integrating, we conclude that

$$\int_{t_3}^t (t-u)^{(m-1)}y'(u)du=(m-1)\int_{t_3}^t (t-u)^{m-2}y(u)du-y(t_3)(t-t_3)^{(m-1)} \tag{16}$$

Thus

$$\int_{t_3}^t\int_a^b (t-u)^{m-1}u^k q(u,\xi)d\sigma(\xi)du \le y(t_3)(t-t_3)^{(m-1)}$$

$$-(m-1)\int_{t_3}^{t}(t-u)^{m-2}y(u)du-\int_{t_3}^{t}(t-u)^{m-1}-(\frac{p(u)}{r(u)}-\frac{k}{u})y(u)du$$

$$-\int_{t_3}^{t}M_1(t-u)^{m-1}\sigma^{(n-2)}(u)\frac{u^{-k}}{r(u)}y^2(u)du$$

$$\frac{1}{t^{m-1}}\int_{t_3}^{t}\int_{a}^{b}(t-u)^{m-1}u^k q(u,\xi)d\sigma(\xi)du\le y(t_3)(\frac{t-t_3}{t})^{m-1}$$

$$-\frac{1}{t^{m-1}}\left\{\int_{t_3}^{t}[(m-1)(t-u)^{m-2}+(t-u)^{m-1}-(\frac{p(u)}{r(u)}-\frac{k}{u})]y(u)du\right.$$

$$\left.+M_1\int_{t_3}^{t}(t-u)^{m-1}\sigma^{(n-2)}(u)\frac{u^{-k}}{r(u)}y^2(u)du\right\}$$

$$=y(t_3)(\frac{t-t_3}{t})^{m-1}-\frac{1}{t^{m-1}}\int_{t_3}^{t}[\sqrt{M_1(t-u)^{m-1}\sigma^{(n-2)}(u)\frac{u^{-k}}{r(u)}}y(u)$$

$$+\frac{(t-u)^{m-2}[(t-u)\frac{p(u)}{r(u)}-\frac{k}{u}+m-1]}{2\sqrt{M_1(t-u)^{m-1}\sigma^{(n-2)}(u)\frac{u^{-k}}{r(u)}}}]^2du$$

$$+\frac{1}{t^{m-1}}\int_{t_3}^{t}\frac{(t-u)^{m-3}u^k[(t-u)\frac{p(u)}{r(u)}-\frac{k}{u}+m-1]^2}{4M_1\frac{1}{r(u)}\sigma^{(n-2)}(u)}du$$

Furthermore, we conclude that

$$\frac{1}{t^{m-1}}\int_{t_3}^{t}(t-u)^{m-3}u^k[\int_{a}^{b}(t-u)^2q(u,\xi)d\sigma(\xi)$$

$$-\frac{[(t-u)\frac{p(u)}{r(u)}-\frac{k}{u}+m-1]^2}{4M_1\frac{1}{r(u)}\sigma^{(n-2)}(u)}]du\le y(t_3)(\frac{t-t_3}{t})^{m-1}$$

Letting $t\to\infty$ in (16), we find that

$$\frac{1}{t^{m-1}}\int_{t_3}^{t}(t-u)^{m-3}u^k[\int_{a}^{b}(t-u)^2q(u,\xi)d\sigma(\xi)$$

$$-\frac{[(t-u)\frac{p(u)}{r(u)}-\frac{k}{u}+m-1]^2}{4M_1\frac{1}{r(u)}\sigma^{(n-2)}(u)}]du\le y(t_3) \tag{17}$$

Which contradicts with (6). This completes the proof of Theorem 1.

Theorem 2. Suppose that

(H7) There exist constant N and α , such that

$$\liminf_{|y|\to\infty}\left|\frac{f(x,y,z)}{z}\right|\ge\alpha>0,x>N$$

(H8) There exists a function $\varphi(\xi)\in C([a,b],(0,\infty))$ such that $\varphi(\xi)\le q(t,\xi).t\ge t_0$, and for any $\xi\in[a,b],\lim_{t\to\infty}q(t,\xi)$ exists.

If for any large T

$$\limsup_{t\to\infty}\frac{1}{t^{m-1}}\int_{T}^{t}\beta(t-u)^{m-3}u^k[(t-u)^2\int_{a}^{b}\varphi(\xi)d\sigma(\xi)$$

$$-\gamma\frac{[(t-u)\frac{p(u)}{r(u)}-\gamma\frac{k}{u}+m-1]^2}{\frac{1}{r(u)}\sigma^{(n-2)}(u)}]du=\infty \tag{18}$$

in which β and γ are some constants, then every solution of equation (1) is oscillatory.

Proof. Suppose that equation (1) has a nonoscillatory solution $x(t)>0$. By using the same arguments as in the proof of Theorem 1, there exists a $t_1 \ge t_0$ such that $x[h(t,\eta)]>0$,

$x[g(t,\eta)]>0, f(x(t)\xi, x[g(t,\xi)]>0$ and $\xi \in [a,b], z^{'}(t)>0, z^{(n-1)}(t)>0$, for $t \ge t_1$, and there exist constant $M>0$ and $t_2 \ge t_1$ such that

$$z^{'}(\frac{t}{2}) \ge Mt^{(n-2)}z^{(n-1)}(t), z^{'}(\frac{\sigma(t)}{2}) \ge M\sigma^{(n-2)}(t)z^{(n-1)}(t) \tag{19}$$

$$\omega(t)=\frac{t^k r(t)z^{(n-1)}(t)}{z[\frac{\sigma(t)}{2}]} \tag{20}$$

Then,

$$\omega^{'}(t)=\frac{t^k(r(t)z^{(n-1)}(t))^{'}}{z[\frac{\sigma(t)}{2}]}+\frac{kt^{k-1}r(t)z^{(n-1)}(t)}{z[\frac{\sigma(t)}{2}]}-\frac{1}{2}\frac{t^k r(t)z^{(n-1)}(t))^{'}}{z^2[\frac{\sigma(t)}{2}]}z^{'}(\frac{\sigma(t)}{2})\sigma^{'}(t)$$

$$\le\frac{t^k(-p(t)z^{(n-1)}(t)-\int_a^b q(t,\xi)f(z(t),\xi,z[g(t,\xi)])d\sigma(\xi)}{z[\frac{\sigma(t)}{2}]}$$

$$+\frac{k}{t}\omega(t)-\frac{1}{2}\frac{t^k r(t)z^{(n-1)}(t)}{z^2[\frac{\sigma(t)}{2}]}z^{'}(\frac{\sigma(t)}{2})\sigma^{'}(t) \tag{21}$$

$$-\frac{1}{2}\frac{t^k r(t)z^{(n-1)}(t)}{z^2[\frac{\sigma(t)}{2}]}z^{'}(\frac{\sigma(t)}{2})\sigma^{'}(t) \tag{22}$$

From $x^{'}(t)>0$, we conclude that $\lim_{t\to\infty} x(t)=L$ exists.

(a) If $L<\infty$, then

$$\liminf_{t\to\infty}\frac{\int_a^b q(t,\xi)f(z(t),\xi,z[g(t,\xi)])d\sigma(\xi)}{z[\frac{\sigma(t)}{2}]}$$

$$\ge\liminf_{t\to\infty}\frac{\int_a^b \varphi(\xi)f(z(t),\xi,z[g(t,\xi)])d\sigma(\xi)}{z[\frac{\sigma(t)}{2}]}$$

$$=\frac{f(L,a,L)}{L}\int_a^b\varphi(\xi)d\sigma(\xi)>0 \tag{23}$$

(b) If $L=\infty$, then we conclude that from *(H2), (H7*) and *(H8*)

$$\liminf_{t\to\infty}\frac{\int_a^b q(t,\xi)f(z(t),\xi,z[g(t,\xi)])d\sigma(\xi)}{z[\frac{\sigma(t)}{2}]}$$

$$\ge\liminf_{t\to\infty}\int_a^b q(t,\xi)\frac{f(z(t),\xi,z[g(t,\xi)])}{z}[\frac{g(t,\xi)}{2}]d\sigma(\xi)$$

$$\geq \liminf_{t\to\infty} \frac{\alpha}{2}\int_a^b \varphi(\xi)d\sigma(\xi) > 0 \tag{24}$$

Let $\beta = \min\left\{(\frac{f(L,a,L)}{L}),(\frac{\alpha}{2})\right\}$, for any large $t_3 \geq t_2$, we conclude that

$$\frac{\int_a^b q(t,\xi) f(z(t),\xi,z[g(t,\xi)])d\sigma(\xi)}{z[\frac{\sigma(t)}{2}]} \geq \beta\int_a^b \varphi(\xi)d\sigma(\xi) \tag{25}$$

Furthermore, from *(H4)*, we conclude that

$$\frac{1}{2}\frac{t^k r(t)z^{(n-1)}(t)}{z^2[\frac{\sigma(t)}{2}]}z^{'}(\frac{\sigma(t)}{2})\sigma^{'}(t) \geq \frac{1}{2}cM\frac{t^{-k}}{r(t)}\sigma^{n-2}(t)\omega^2(t) \tag{26}$$

in which $M_2 = \frac{1}{2}Mc$, then

$$\omega^{'}(t) \leq -\beta t^k\int_a^b \varphi(\xi)d\sigma(\xi) - (\frac{p(t)}{r(t)} - \frac{k}{t})\omega(t) - \frac{1}{2}M_2\frac{t^{-k}}{r(t)}\sigma^{n-2}\frac{t^{-k}}{r(t)}\omega^2(t) \tag{27}$$

The remainder proof is as same as proof of Theorem 1, we omit it. This completes the proof of Theorem 2.

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