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Existence and Uniqueness for Backward Stochastic Differential Equation to Stopping Time*

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Abstract

In this paper, we prove the existence and uniqueness for Backward Stochastic Differential Equations with stopping time as time horizon under the hypothesis that the generator is bounded. We first prove for the stopping time with finite values and for the general stopping time we prove the result taking limit. We suggest a new approach to generalize the results for the case of constant time horizon to the case of stopping time horizon.

Keywords: BSDE (Backward Stochastic Differential Equation); Random Time Horizon; Stopping Time

1 INTRODUCTION

Linear backward stochastic differential equations (BSDEs) were introduced by Bismut[1] as the adjoint equations associated with stochastic Pontryagin maximum principles in stochastic control theory. The general case of non-linear BSDEs was then studied by Pardoux and Peng. (see [2] and [3] in the Brownian framework). In [3], they provided Feynman-Kac representations of solutions of non-linear parabolic partial differential equations (PDEs). In the paper by El Karoui et al.[4], some additional properties are given and several applications to option pricing and recursive utilities are studied.

The case of a discontinuous framework is more involved, especially concerning the comparison theorem, which requires an additional assumption. In 1994, Tang and Li[5] provided an existence and uniqueness result in the case of a natural filtration associated with a Brownian motion and a Poisson random measure. In 1995, Barles, Buckdahn and Pardoux[6] provided a comparison theorem as well as some links between BSDEs and non-linear parabolic integral-PDEs, generalizing some results of [2] to the case of jumps. In 2006, Royer[7] proved a comparison theorem under weaker assumptions, and introduced the notion of non-linear expectations in this framework.

Since then BSDEs have been widely used in mathematical finance and partial differential equations (PDEs). Many pricing problems can be written in terms of linear BSDEs or non-linear BSDEs when portfolios constraints are taken into account (see, e.g., El Karoui et al.[4], A. Bensoussan[8], I. Karatzas[9], and B. Wang and Q. Meng[10],). And numerous results show the intimate relationship between BSDEs and PDEs, which suggests that existence and uniqueness results which can be obtained on one side should have their counterparts on the other side.

Many mathematician have worked to improve the existence / uniqueness condition of a solution for BSDEs in connection with the specific applications.

Most of those works are concerned with the case of constant time horizon. But in many applications we encounter the case of random time horizon. For example, Marcus and V´eron[11] show that the solutions to $PDE-\Delta \mathrm{u}+\mathrm{u}\left|\mathrm{u}\right|^{\mathrm{q}}=0$ are related to the BSDE $Y_t=\xi-\int_t^{\tau} Y_r\left|Y_r\right|^q dr-\int_t^{\tau} Z_r dW_r$. Here the time horizon τ is a

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stopping time. And in the finance, there are many cases when the time horizon is not constant but random. There are many research papers about the case of random time horizon in connections with applications[12].

But those works on the case of random time horizon were limited to their specific settings and there hasn't been a general approach to deal with BSDEs to stopping time horizon. A stopping time is a very special random variable and many work shows that it can be treated as a constant. (Doob's Optional Sampling Theorem is a very nice example.)

In this paper, we prove the existence and uniqueness for BSDEs to stopping time horizon. Actually we suggest a new method to generalize the results for the case of constant time horizon to the case of stopping time horizon. We note that our method is very simple and clear, and it can be used in many applications.

The paper is organized as follows. In section 2, we quote existence and uniqueness results for the case of constant time horizon. In Section 3, we prove the existence and uniqueness for the case of finite value stopping time horizon. In Section 4, we generalize the existence and uniqueness to the case of bounded stopping time horizon.

2 TO CONSTANT TIME HORIZON

In this section we quote existence and uniqueness results for the case of constant time horizon. We follow the terminology in [13].

Let $(\Omega, \mathcal{F}, P)$ be a probability space on which is defined a d-dimensional Brownian motion $W := (W_t : t < T)$. Let us denote by $(\mathcal{F}_t^W : t < T)$ the natural filtration of W and $(\mathcal{F}_t^W : t < T)$ is completion with the P -null sets of $\mathcal{F}$. We define the followiing spaces:

- $\mathcal{P}_n$ the set of $\mathcal{F}_t$-progressively measurable, $\mathcal{R}^n$-valued processes on $\Omega \times [0,T)$
- $\mathcal{L}_n^2(\mathcal{F}_t) := \left\{\eta : \mathcal{F}_t\text{-measurable random?}\mathcal{R}^n\text{-valued variable s.t. } E\left(|\eta|^2\right) < \infty\right\}$
- $S_n^2(0,T) := \left\{\varphi \in \mathcal{P}_n \text{with continuous paths, s.t.} E\left(\sup_{t<T} |\varphi_t|^2\right) < \infty\right\}$
- $\mathcal{H}_n^2 = \{Z \in \mathcal{P}_n \text{ s.t. } E\left(\int_0^T |Z_s|^2 ds\right) < \infty\}$
- $\mathcal{H}_n^1 := \{Z \in \mathcal{P}_n \text{ s.t. } E\left[\left(\int_0^T |Z_s|^2 ds\right)^{\frac{1}{2}}\right] < \infty\}$

Let us now introduce the notion of multi-dimensional BSDE.

Definition 2.1. Let $\xi \in \mathcal{L}_n^2(\mathcal{F}_T)$ be a R^m -valued terminal condition and let g be a R^n -valued generator, $\mathcal{P}_m \otimes \mathcal{B}\left(R^m \times R^{m\times d}\right)$-measurable. A solution for the m-dimensional BSDE associated with parameters (g,ξ) is a pair of progressively measurable processes $Y \in S_m^2(0,T), Z \in \mathcal{H}_{m\times d}^2(0,T)$ such that

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s)\, ds - \int_t^T Z_s dW_s, \qquad 0 \le t \le T \tag{2.1}$$

The differential form of this equation is

$$-dY_t = g(t, Y_t, Z_t)\,dt - Z_t dW_t, Y_T = \xi.$$

Hereafter g is called the generator and ξ the terminal value of the BSDE.

Under some specific assumptions on the generator g , the BSDE (2.1) has a unique solution. The standard assumptions are the following:

(i) $\left(g(t,0,0): t\le T\right)\in \mathcal{H}_m^2$;

(ii) g is uniformly Lipschitz with respect to (y,z): there exists a constant $C\ge 0$ such that

$$\left|g(\omega,t,y,z)-g(\omega,t,y',z')\right|\le C\left(|y-y'|+|z-z'|\right) \quad for\ \forall (y,y',z,z')$$

Theorem 2.1 (By Pardoux and Peng [2]). Under the above standard assumptions (i) and (ii), there exists a unique solution (Y,Z) of the BSDE (2.1) with parameters (g,ξ) .

3 TO FINITE VALUE STOPPING TIME HORIZON

In this section we prove the existence and uniqueness for the case of finite value stopping time horizon.

Let τ be a $(\mathcal{F}_t)$-stopping time that takes finite number of values. Without loss of generality assume that $\tau > 0$. The following lemma shows the condition for a random variable with finite values to be a stopping time.

Lemma 3.1. Let τ be a positive random variable on $(\Omega,\mathcal{F},P)$ with finite values. That is

$$\tau := \sum_{i=1}^{n} a_i\cdot\chi(A_i), \qquad 0<a_i<\cdots<a_n<\infty$$

Then τ is a $(\mathcal{F}_t)$-stopping time if and only if for any $k, A_k\in\mathcal{F}_{a_k}$ and $\sum_{i=k+1}^{n} A_i\in\mathcal{F}_{a_k}$.

Proof. $(\Rightarrow)$ As τ is a stopping time, for $\forall k>1$

$$\{\tau\le a_k\}=\sum_{i=1}^{k} A_i\in\mathcal{F}_{a_k} \quad \text{and} \quad \{\tau\le a_{k-1}\}\sum_{i=1}^{k-1} A_i\in\mathcal{F}_{a_{k-1}}\subset\mathcal{F}_{a_k}$$

So, $A_k=\sum_{i=1}^{k} A_i\backslash\sum_{i=1}^{k-1} A_i\in\mathcal{F}_{a_k}$. And $\sum_{i=1}^{k} A_i\in\mathcal{F}_{a_k}$ implies that $\sum_{i=k+1}^{n} A_i=\Omega\backslash\sum_{i=1}^{k} A_i\in\mathcal{F}_{a_k}$.

$(\Leftarrow)$ Clear.

And furthermore $A\in\mathcal{F}_\tau \Leftrightarrow \forall k, A\cap A_k\in\mathcal{F}_{a_k}$. In fact $\forall k, A\bigcap\{\tau\le a_k\}\in\mathcal{F}_{a_k}$ implies

$$A\bigcap\{\tau=a_k\}=\left(A\bigcap\{\tau\le a_k\}\right)\backslash\left(A\bigcap\{\tau\le a_{k-1}\}\right)=A\bigcap A_k\in\mathcal{F}_{a_k}$$

and the inverse is clear.

Now consider a BSDE to a stopping time horizon with finite values.

$$Y_t=\xi+\int_t^\tau g(s,Y_s,Z_s)ds-\int_t^\tau Z_s dW_s, \qquad 0\le t\le\tau. \tag{3.2}$$

Theorem 3.1. Let τ be a $(\mathcal{F}_t)$-stopping time that takes finite number of values and $\xi\in\mathcal{L}_M^2(\mathcal{F}_\tau)$, $g:[0,\tau]\times R^m\times R^{m\times d}\to R^m$ be $\mathcal{P}_m\otimes\mathcal{B}\left(R^m\times R^{m\times d}\right)$-measurable. Then under the assumptions

(i) $\left(g(t,0,0): t\le\tau\right)\in\mathcal{H}_m^2$

(ii) g is uniformly Lipschitz with respect to (y,z): there exists a constant $C\ge 0$ such that

$$\left|g(\omega,t,y,z)-g(\omega,t,y',z')\right|\le C\left(|y-y'|+|z-z'|\right) \quad \text{for}\ \forall (y,y',z,z')$$

there exists a unique solution (Y,Z) of the BSDE (3.2).

Proof. Let $\tau=\sum_{i=1}^{n} a_i\cdot\chi(A_i)$,where $0<a_i<\cdots<a_n<\infty$ .If we set $a_0=0$, $B_k=\sum_{i=k}^{n} A_i$ and $E_k=B_k\times(a_{k-1},a_k]$, then $B_k\in\mathcal{F}_{a_{k-1}}$ by the Lemma 3.1. And

$$\chi(s\le\tau)=\sum_{i=1}^{n}\chi(s\le a_i)\cdot\chi(A_i)=\sum_{i=1}^{n}\chi(a_{i-1}<s\le a_i)\cdot\chi(B_i)\quad (\text{see Figure 1}).$$

Then

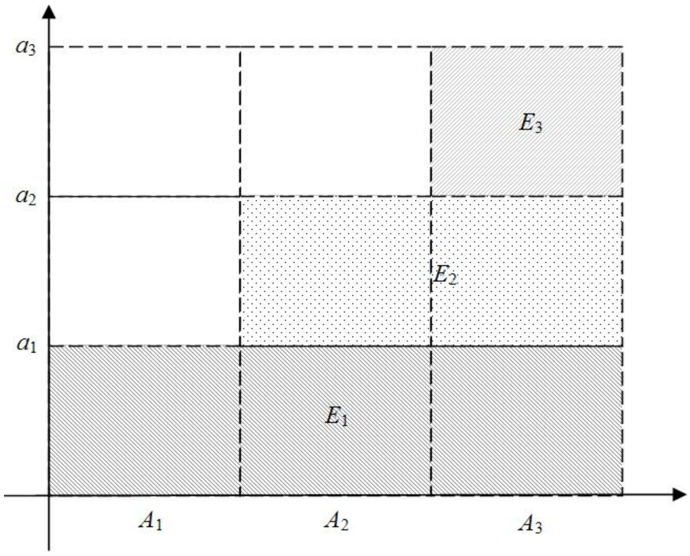

FIGURE 1: DOMAIN OF BSDE

$$\begin{aligned}\int_t^\tau Z_s dW_s &= \int_t^\infty \sum_{i=1}^{n}\chi(a_{i-1}<s\le a_i)\cdot\chi(B_i)\cdot Z_s dW_s\\ &=\sum_{i=1}^{n}\int_t^\infty \chi(a_{i-1}<s\le a_i)\cdot\chi(B_i)\cdot Z_s dW_s\\ &=\sum_{i=1}^{n}\int_{t\vee a_{i-1}}^{t\vee a_i}\chi(B_i)\cdot Z_s dW_s\end{aligned}$$

Likewise

$$\int_t^\tau g(s,Y_s,Z_s)ds == \sum_{i=1}^{n}\int_{t\vee a_{i-1}}^{t\vee a_i}\chi(B_i)\cdot g(s,Y_s,Z_s)dW_s\ .$$

So the original equation can be written as

$$Y_t=\sum_{i=1}^{n}\left[\xi\cdot\chi(A_i)+\int_{t\vee a_{i-1}}^{t\vee a_i}\chi(B_i)\cdot g(s,Y_s,Z_s)dW_s-\int_{t\vee a_{i-1}}^{t\vee a_i}\chi(B_i)\cdot Z_s dW_s\right]$$

Now we "contract" the space as following. Consider B_k as a space and define $\sigma-\text{algebra}$ on B_k by $\mathcal{F}^k:\{A\cap B_k \mid \mathrm{A}\in\mathcal{F}\}$. And define the probability measure by $P^k\left(A\cap B^k\right):=\frac{P(A)}{P(B^k)}$ and filter by

$$\mathcal{F}_t^k:\{A\cap B_k\mid \mathrm{A}\in\mathcal{F}_t\},\qquad \mathrm{F}^k:=\left\{\mathcal{F}_t^k: t\in(a_{k-1},a_k]\right\}\ .$$

These elements are all well-defined and we get a new basis $\left(B_k,P^k,\mathcal{F}^k,\mathrm{F}^k\right)$. If we set

$$W_t^k(\omega):=W_t(\omega)-W_{a_{k-1}}(\omega),\qquad \omega\in B_k,\quad t\in(a_{k-1},a_k],$$

$W_t^k(\omega)$ becomes a d -dimensional Brownian motion on the new basis. In fact

- $W^k_{a_{k-1}} = 0$;
- If $a_k \ge t > s > a_{k-1}$, then

$$\int_A \left(W_t^k - W_s^k\right) dP^k = \int_{B_k \cap B} \left(W_t - W_s\right) dP \cdot \frac{1}{P\left(B_k\right)} = 0, \quad \forall A \in \mathcal{F}_s^k \ ,$$

Where $B_k \in \mathcal{F}_{a_{k-1}} \subset \mathcal{F}_s, B \in \mathcal{F}_s$. So

$$E^k\left(W_t^k \mid \mathcal{F}_s^k\right) = W_s^k \qquad \text{a.s.},$$

Where E^k means an expectation under P^k ;

- $\left(W_t - W_s\right) \sim \mathcal{N}\left(0, t-s\right)$ and for $\forall c \in R$,

$$P^k\left(W_t^k - W_s^k \le c\right) = \frac{P^k\left(\left\{W_t^k - W_s^k \le c\right\} \cap B_k\right)}{P\left(B_k\right)} = \frac{P^k\left(\left\{W_t^k - W_s^k \le c\right\}\right) \cap P\left(B_k\right)}{P\left(B_k\right)}$$
$$= P^k\left(\left\{W_t^k - W_s^k \le c\right\}\right);$$

Now we define n BSDEs on new bases and verify the condition for existence and uniqueness.

- On $\left(B_k, P^k, \mathcal{F}^k, \mathrm{F}^k\right)$. Set $\xi^n := \xi \cdot \chi\left(A_n\right)$ and consider

$$Y_t^n = \xi_n + \int_t^{a_n} g\left(s, Y_s^n, Z_s^n\right) ds - \int_{\mathrm{t}}^{a_n} Z_s^n dW_s^n, \quad \mathrm{a}_{n-1} \le t \le a_n .$$

Then ξ^n is $\mathcal{F}_{a_n}^n$ -measurable and square integrable. And g satisfies conditions for existence and uniqueness for the case of constant time horizon.

- On $\left(B_k, P^k, \mathcal{F}^k, \mathrm{F}^k\right), k < n$. Set

$$\xi^k := \begin{cases} \xi \cdot \chi\left(A_k\right), \omega \in A_k; \\ Y_{a_k}^{k+1}, \omega \in B_k \setminus A_k = B_{k+1}. \end{cases}$$

and consider

$$Y_t^k = \xi_k + \int_t^{a_k} g\left(s, Y_s^k, Z_s^k\right) ds - \int_{\mathrm{t}}^{a_k} Z_s^k dW_s^k, \quad \mathrm{a}_{k-1} \le t \le a_k .$$

Then $\xi^k \in \mathcal{L}_m^2\left(\mathcal{F}_{a_k}^k\right)$. In fact $\left\{\xi^k \le t\right\} = \left(A_k \cap \left\{\xi \le t\right\}\right) \cup \left\{Y_{a_k+}^{k+1} \le t\right\}$ and $A_k \cap \left\{\xi \le t\right\} \in \mathcal{F}_{a_k}^k \left(\xi \in \mathcal{F}_t\right)$.

So $A_k \cap \left\{\xi \le t\right\} = A_k \cap \left(A_k \cap \left\{\xi \le t\right\}\right) \in \mathcal{F}_{a_k}^k$. And $\left\{Y_{a_k+}^{k+1} \le t\right\} = \cap_{l \in N} \left\{Y_{a_{k+\frac{1}{l}}}^{k+1} \le t\right\} \in \mathcal{F}_{a_k+}^{k+1} = \mathcal{F}_{a_k}^{k+1} \subset \mathcal{F}_{a_k}^k$ shows that ξ^k is $\mathcal{F}_{a_k}^k$ -measurable. On the other hand.

$$\int_{B_k} \left(\xi^k\right)^2 P^k\left(dw\right) = \int_{A_k} \xi^2 P^k\left(dw\right) + \int_{B_k} \left(Y_{a_k+}^{k+1}\right)^2 P^k\left(dw\right)$$
$$< \int_\Omega \xi^2 P^k\left(dw\right) \frac{1}{P\left(B_k\right)} + \int_{B_{k+1}} \left(Y_{a_k+}^{k+1}\right)^2 P^{k+1}\left(dw\right) \frac{P\left(B_{k+1}\right)}{P\left(B_k\right)} < \infty$$

So $\xi^k \in \mathcal{L}_m^2\left(\mathcal{F}_{a_k}^k\right)$ and similarly one can verify that g satisfies conditions for existence and uniqueness for the case of constant time horizon.

Now we combine the solutions $\left\{Y_t^k, \mathrm{Z}_t^k\right\}$ and set

$$Y(t,\omega) := \sum_{k=1}^{n} Y_t^k(\omega)\cdot\chi(E_k) + \xi\cdot\chi\left(\left(\sum_{k=1}^{n} E_k\right)^c\right),$$

$$Z(t,\omega) := \sum_{k=1}^{n} Z_t^k(\omega)\cdot\chi(E_k).$$

Then (Y,Z) becomes a unique solution of the BSDE(3.2). In fact

- For any t, let the interval containing it be $(a_{k-1}, a_k]$, then

$$\{Y_t \le c\} = (\{Y_t \le c\}\cap B_k)\cup(\{Y_t \le c\}\cap B_k^c) = (\{Y_t \le c\}\cap B_k)\cup(\{\xi \le c\}\cap B_k^c)\in \mathcal{F}_t^k, \quad \forall c\in R.$$

- We can write

$$Y_t = \begin{cases} \xi, & \omega\in B_k^c; \\ \xi + \int_t^{a_k} g(s,Y_s^k,Z_s^k)ds - \int_t^{a_k} Z_s^k dW_s^k, & \omega\in A_k; \\ Y_{a_k}^{k+1} + \int_t^{a_k} g(s,Y_s^k,Z_s^k)ds - \int_t^{a_k} Z_s^k dW_s^k, & \omega\in B_k\setminus A_k. \end{cases}$$

So using $Y_{a_k}^{k+1} = \xi^k + \int_{a_k}^{a_{k+1}} g(s,Y_s^{k+1},Z_s^{k+1})ds - \int_{a_k}^{a_{k+1}} Z_s^{k+1} dW_s^{k+1}$, we can easily check that

$$Y_t = \xi + \int_t^{\tau} g(s,Y_s,Z_s)ds - \int_t^{\tau} Z_s dW_s$$

If the equation(3.2)has another solution, its restriction to E_k becomes a solution on $(B_k, P^k, \mathcal{F}^k, \mathrm{F}^k)$ and this coincides with (Y_t^k, Z_t^k).

4 TO BOUNDED STOPPING TIME

In this section we generalize the existence and uniqueness to the case of bounded stopping time horizon.

First we show a lemma which is useful for argument of limitation

Lemma 4.1. Let $\xi\in\mathcal{L}_M^2(\mathcal{F}_\tau)$, $g:\Omega\times[0,T]\times R^m\times R^{m\times d}\to R^m$ be $\mathcal{P}_m\otimes\mathcal{B}(R^m\times R^{m\times d})$-measurable and be a solution of the following BSDE to τ :

$$Y_t = \xi + \int_t^{\tau} g(s,Y_s,Z_s)ds - \int_t^{\tau} Z_s dW_s, \quad 0\le t\le\tau.$$

Then for any positive real number β, we have the following inequality.

$$\left|Y_{t_1} - Y_{t_2}\right|^2 \le E^{\mathcal{F}_t}\left(\int_{t_1}^{t^2} \frac{2}{\beta}\left|g(\mathrm{s},\mathrm{Y}_s,\mathrm{Z}_s)\right|^2 e^{\beta(s-t_1)}ds\right) \tag{4.3}$$

Especially (Yt) is almost continuous in t.

Proof. Just from the equation

$$\begin{aligned} Y_{t_1} &= \xi + \int_{t_1}^{\tau} g(s,Y_s,Z_s)ds - \int_{t_1}^{\tau} Z_s dW_s \\ &= \xi + \int_{t_2}^{\tau} g(s,Y_s,Z_s)ds - \int_{t_2}^{\tau} Z_s dW_s + \int_{t_1}^{t_2} g(s,Y_s,Z_s)ds - \int_{t_1}^{t_2} Z_s dW_s \\ &= Y_{t_2} + \int_{t_1}^{t_2} g(s,Y_s,Z_s)ds - \int_{t_1}^{t_2} Z_s dW_s, \qquad \forall t_1 < t_2, \end{aligned}$$

and we can write for all $t\in(t_1,t_2), Y_t = Y_{t_2} + \int_t^{t_2} g(s,Y_s,Z_s)\,ds - \int_t^{t_2} Z_s dW_s$. Now we apply Ito's formula to $\left|Y_{t_1}-Y_{t_2}\right|^2 e^{\beta(s-t_1)}$ in $[t_1,t_2]$ and we get

$$-\left|Y_{t_1}-Y_{t_2}\right|^2 = \int_{t_2}^{t_1}\left[-2\left(Y_{t_1}-Y_{t_2}\right)g(s,Y_s,Z_s)+\left|Z_s\right|^2+\beta\left|Y_{t_1}-Y_{t_2}\right|^2\right]e^{\beta(s-t_1)}ds$$
$$+\int_{t_2}^{t_1} 2\left(Y_{t_1}-Y_{t_2}\right)Z_s e^{\beta(s-t_1)}dW_s,$$

taking conditional expectation with respect to $\mathcal{F}_{t_1}$ in both sides

$$\left|Y_{t_1}-Y_{t_2}\right|^2 + e^{\mathcal{F}_t}\left(\int_{t_2}^{t_1}\left(\beta\left|Y_{t_1}-Y_{t_2}\right|^2+\left|Z_s\right|^2\right)e^{\beta(s-t_1)}dt\right) = e^{\mathcal{F}_t}\left(\int_{t_2}^{t_1}2\left(Y_s-Y_{t_2}\right)g(s,Y_s,Z_s)\,ds\right)$$
$$\le e^{\mathcal{F}_t}\left(\int_{t_2}^{t_1}\left(\frac{\beta}{2}\left|Y_{t_1}-Y_{t_2}\right|^2+\frac{2}{\beta}\left|g(s,Y_s,Z_s)\right|^2\right)e^{\beta(s-t_1)}ds\right),$$

So we get

$$\left|Y_{t_1}-Y_{t_2}\right|^2 \le E^{\mathcal{F}_t}\left(\int_{t_1}^{t^2}\frac{2}{\beta}\left|g(\mathrm{s},\mathrm{Y}_s,\mathrm{Z}_s)\right|^2 e^{\beta(s-t_1)}ds\right)$$

and it proves the first assertion.

And because g and $e^{\beta(s-t_1)}$ are both bounded in $[t_1,t_2]$, taking expectation in both sides, we get $\exists D>0$, $E\left(\left|Y_{t_1}-Y_{t_2}\right|^2\right)\le D\cdot\left|t_2-t_1\right|^2$, and by Kolmogorov's continuity theorem there exists a continuous version of (Y_t) .

Now we state the main result of this paper.

Theorem 4.1. Let τ be a bounded $(\mathcal{F}_t)$ -stopping time and $\xi\in\mathcal{L}_M^2(\mathcal{F}_\tau)$, $g:[0,\tau]\times R^m\times R^{m\times d}\to R^m$ be $\mathcal{P}_m\otimes\mathcal{B}\left(R^m\times R^{m\times d}\right)$ -measurable and bounded. Then under the assumptions

(i) $\left(g(t,0,0):t\le\tau\right)\in\mathcal{H}_m^2$

(ii) g is uniformly Lipschitz with respect to (y,z) : there exists a constant $C\ge 0$ such that

$$\left|g(\omega,t,y,z)-g(\omega,t,y',z')\right|\le C\left(\left|y-y'\right|+\left|z-z'\right|\right)\quad for\ \forall(y,y',z,z')$$

there exists a unique solution (Y,Z) of the BSDE

$$Y_t=\xi+\int_t^\tau g(s,Y_s,Z_s)\,ds-\int_{\mathrm{t}}^\tau Z_s dW_s,\quad 0\le t\le\tau.$$

Proof. Let (τ_n) be a decreasing sequence which converges to τ . Actually for bounded stopping time τ , we can always take

$$\tau_n=\sum_{k=0}^{\infty}\frac{k+1}{2^n}\chi\left(\frac{k}{2^n}\le\tau\le\frac{k+1}{2^n}\right)$$

and it's clear that this is a sequence of stopping times.

For each n , from Theorem 3.1 there exists a unique solution $\left(Y_t^n,Z_t^n\right)$ of the equation

$$Y_t^n=\xi+\int_t^{\tau_n} g\left(s,Y_s^n,Z_s^n\right)ds-\int_{\mathrm{t}}^{\tau_n} Z_s^n dW_s$$

then

$$
\begin{aligned}
Y_t^n &= \xi + \int_t^{\tau_n} g\left(s, Y_s^n, Z_s^n\right) ds - \int_t^{\tau_n} Z_s^n dW_s \\
&= \xi + \int_\tau^{\tau_n} g\left(s, Y_s^n, Z_s^n\right) ds - \int_\tau^{\tau_n} Z_s^n dW_s + \int_t^{\tau} g\left(s, Y_s^n, Z_s^n\right) ds - \int_t^{\tau} Z_s^n dW_s \\
&= Y_\tau^n + \int_t^{\tau} g\left(s, Y_s^n, Z_s^n\right) ds - \int_t^{\tau} Z_s^n dW_s,
\end{aligned}
$$

and $\left(Y_t^n, Z_t^n\right)$ can be considered as a solution of BSDE with Y_τ^n as a terminal variable. (Note that $\xi \in \mathcal{L}_M^2\left(\mathcal{F}_\tau\right) \subset \mathcal{L}_M^2\left(\mathcal{F}_{\tau_n}\right)$), so for any $m > n$,

$$
Y_t^m - Y_t^n = Y_\tau^m - Y_\tau^n + \int_t^{\tau} \left(g\left(s, Y_s^m, Z_s^m\right) - g\left(s, Y_s^n, Z_s^n\right)\right) ds - \int_t^{\tau} \left(Z_s^m - Z_s^n\right) dW_s
$$

Now we apply the famous Ito's formula to $\left|Y_s^m - Y_s^n\right|^2 e^{\beta(s-t)}$ in $[t, \tau]$, and we get

$$
\begin{aligned}
&\left|Y_\tau^m - Y_\tau^n\right|^2 e^{\beta(\tau-t)} - \left|Y_t^m - Y_t^n\right|^2 \\
&= \int_t^\tau \left[-2\left(Y_s^m - Y_s^n\right)\left(g^m - g^n\right) + \beta\left|Y_s^m - Y_s^n\right|^2 + \left|Z_s^m - Z_s^n\right|^2\right] e^{\beta(s-t)} ds \\
&+ 2\int_t^\tau e^{\beta(s-t)}\left(Y_s^m - Y_s^n\right)\left(Z_s^m - Z_s^n\right) dW_s
\end{aligned}
$$

where $g^m = g\left(s, Y_s^m, Z_s^m\right), g^n = g\left(s, Y_s^n, Z_s^n\right)$. Multiplying $e^{\beta\tau}$ to the both sides and taking expectation, we get

$$
\begin{aligned}
&e^{\beta t} E\left|Y_t^m - Y_t^n\right|^2 + E\left(\int_t^\tau \left(\beta\left|Y_s^m - Y_s^n\right|^2 + \left|Z_s^m - Z_s^n\right|^2\right) e^{\beta s} ds\right) \\
&= E\left(\left|Y_\tau^m - Y_\tau^n\right|^2 e^{\beta\tau}\right) + E\left(\int_t^\tau 2\left(Y_s^m - Y_s^n\right)\left(g^m - g^n\right) e^{\beta s} ds\right) \\
&\le E\left(\left|Y_\tau^m - Y_\tau^n\right|^2 e^{\beta\tau}\right) + E\left(\int_t^\tau \left(\frac{\beta}{2}\left|Y_s^m - Y_s^n\right|^2 + \frac{2}{\beta}\left|g^m - g^n\right|^2\right) e^{\beta s} ds\right) \\
&\le E\left(\left|Y_\tau^m - Y_\tau^n\right|^2 e^{\beta\tau}\right) + E\left(\int_t^\tau \left(\frac{\beta}{2}\left|Y_s^m - Y_s^n\right|^2 + \frac{4C^2}{\beta}\left(\left|Y_s^m - Y_s^n\right|^2 + \left|Z_s^m - Z_s^n\right|^2\right)\right) e^{\beta s} ds\right)
\end{aligned}
$$

If we set $\beta_0 = 16\left(1 + C^2\right)$, the above inequality implies the following one.

$$
e^{\beta_0 t} E\left|Y_t^m - Y_t^n\right|^2 + \frac{1}{4} E\int_t^\tau \left(\left|Y_s^m - Y_s^n\right|^2 + \left|Z_s^m - Z_s^n\right|^2\right) e^{\beta_0 s} ds \le E\left(\left|Y_\tau^m - Y_\tau^n\right|^2 e^{\beta_0\tau}\right)
$$

On the other hand, the right side of the above inequality satisfies

$$
E\left(\left|Y_\tau^m - Y_\tau^n\right|^2 e^{\beta_0\tau}\right) = E\left(\left|Y_\tau^m - \xi + \xi - Y_\tau^n\right|^2 e^{\beta_0\tau}\right) \le 2E\left(\left(\left|Y_\tau^m - Y_{\tau_m}^m\right|^2 + \left|Y_\tau^n - Y_{\tau_n}^n\right|^2\right) e^{\beta_0\tau}\right)
$$

by the Lemma 4.1 and it gets very small as $n \to \infty$ because of the continuity of solution.

So $\left(Y_t^n, Z_t^n\right)$ becomes a Cauchy sequence in the space $\mathcal{H}_{m\times(d+1)}^2\left(0, \tau\right)$ equipped with a norm

$$
\left\|X(\bullet)\right\|_\beta := \left\{E\int_0^\tau \left|X_s\right|^2 e^{\beta s} ds\right\}^{\frac{1}{2}}
$$

and converges to certain pair $\left(Y_t, Z_t\right)$. And it's clear that this becomes the unique solution of our main equation.

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