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Indifference Pricing in the Single Period Binomial with Complete Market Model

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Abstract

Binomial no-arbitrage price have a method is the traditional approach for derivative pricing, which is, the complete model, which makes possible the perfect replication in the market. Risk neutral pricing is an appropriate method of asset pricing in a complete market. We have discussed an incomplete market, a non - transaction asset that produces incompleteness of the market. An effective method of asset pricing in incomplete markets is the undifferentiated pricing method. This technique was firstly introduced by Bernoulli in (1738) the sense of gambling, lottery and their expected return. It is used to command investors' preferences and better returns the results they expect. In addition, we also discuss the utility function, which is the core element of the undifferentiated pricing. We also studied some important behavior preferences of agents, and injected exponential effect of risk aversion in the model, so that the model was nonlinear in the process of claim settlement.

Keywords: Complete Market Model; Option Pricing; Nonlinear Pricing Formula; Risk Natural Measure; Expected Utility and Indifference Pricing

1 INTRODUCTION

The fundamental principle of pricing theory is that there is no arbitrage opportunity in the ideal financial market. In the real world, arbitrage opportunities do exist, but only in a short period of time. In the pricing of derivatives, we usually distinguish between complete and incomplete markets. A market is called complete, if every request can be perfectly copied, that is to say, investors can establish a portfolio at zero time and have a suitable trading strategy, which makes it possible to reproduce the benefits of mussel when the time is ripe. In a complete market with no arbitrage hypothesis, the price of any claim on the market is uniquely determined as the value of its duplicated portfolio. A simple example of a complete market (see Steven E. Sharive “Calculus I" Chapter I) includes the one period binomial model with a risky asset and a money market account, where the rate for interest is zero. A simple example of a continuous time model is the standard Black - Scholes model, and its bonds and risky asset stocks are modelled as diffusion. Every time someone wants to copy a claim perfectly, there are two ways to find its value. First, one can copy the related replication strategy with the number of tradable asset units required for zero time and maturity. Using the recursion of replication strategy at the price of zero time and tradable assets, it can then easily calculate the value of the duplicated portfolio. Secondly, those who only want to price claims, rather than copy it, can ignore things and apply the basic theorem of asset pricing (FATP) instead of the first mentioned method. One of the important statements of the (FATP) is that under the nonexistence of arbitrage, there exists measure which is equivalent to the real world measure. Under this assumption, all tradable asset markets in the discounted price process are martingales. As an important formula, the value of the claim can be recursively calculated in the backward time as (1.1)

$$C(0)=E_{\mathbb{Q}}\left(C(T)\right) \tag{1.1}$$

where, for the sake of simplicity, we assume that the interest rate is constant. The interesting matter for a complete market is that the equivalent martingale measure is unique, the price of the claim is uniquely determined by the formula (1:1), and it's also called the law of one price. The whole market is a frictionless market, so it is possible to borrow money from the money market account at the same rate at the same loan.

A market is called incomplete market, and certain claims can not be completely hedged. The measures to price derivatives in incomplete markets are no longer the only one. The claim price cannot be determined by the law of one price. An example of an incomplete market is the undifferentiated pricing in a single period binomial model. By definition, any binomial model contains a non - negotiable asset that makes the market incomplete. Most markets are not entirely due to nontradability assets or market frictions. The complete market is the ideal approximation of the incomplete market. In incomplete markets, derivative pricing is not the only method. Since this not unique, the investor has to make a decision to choose which measure is appropriate. The investor's choice for appropriatedepends on his risk preference. Pricing in incomplete markets is difficult. There are many methods, such as super replicas, minimal martingales, and convex risk measures. Super duplication is a portfolio that determines or exceeds the maturity of the time to a certain extent. A powerful pricing method in incomplete markets, the utility undifferentiated pricing method. It is derived from the practicality of actuarial mathematics. The advantage of utility indifference pricing is that it carries out an economic argument, and it relies on exponential utility to make the price nonlinear. Exponential utility has the ability to calculate the initial wealth and endowment, which makes mathematics easy to understand. In indiscriminant pricing, most people prefer more wealth rather than less wealth. A person can maximize his relative expectation of money endowments. The apathy pricing method is the relativity between the content of Bernoulli (1738) and the calculation of a person's expected utility evaluation instead of the currency outlook. This method has the logical consistency of standards and economic rationality, by Von Neumann and Morgenstern (1944) based on the structure of investors preference, and expand the Barbarians (1972), no difference in pricing is the risk appetite of the agent's expected utility representation. The rest of this arrangement, Section 2: a complete phase model, the 3 part: Discussion of incomplete phase binomial model and nonlinear price formula, the 4 and the 5 section is the main part of this paper, the 4 part is about the convex risk measure, the 5 part is about the two limit price behavior and their indifference with the expected utility, the last part is the conclusion.

2 COMPLETE ONE PERIOD MODEL

2.1 The Extraction of Emotional Information

A complete periodic model is a periodic binomial model. Simply speaking , the model is consist of two tradable assets; a riskless asset, which is money market account, called bond, denoted by B, and a risky asset, called stock, denoted by S . The two assets are traded at times 0 and at maturity T . This key has a unit initial value, with a constant interest rate. The horizon for time $\left[0,T\right]$, the riskless asset, costs $B_T = 1+r$. But in this model, for the sake of simplicity, we assume that the interest rate is $r=0$ so that the bond cost at maturity is still $B_T = 1$.

Probabilistic spatial description of the randomness of risk assets $\left(\Omega,\mathcal{F}_T,P\right)$, where the random elements from $\Omega=\left(\omega_1,\omega_2\right)$. The branching probability for the model are $P:=P\left(\omega_1\right)>0$ and $\mathrm{q}:=1-p=P\left(\omega_2\right)$, and $\mathcal{F}_T = 2^{\Omega}$. S_T is random variables on Ω , $S_T = S_0\xi$ with

$$\xi\left(\omega\right)=\begin{cases} u, \text{if } \omega=\omega_1; \\ d, \text{if } \omega=\omega_2; \end{cases}$$

and the $0<d<1<u$ condition for arbitrage-free should be satisfied.

Let C be a contingent claim written on S with payoff C_T . We have a European call option written on the asset S with an expiration data T and strike price K. One of the pricing methods of this model is replicating. Each claim can be copied into a complete model. In fact, we have a function of $\varphi=\left(\alpha,\beta\right)$ is the portfolio of this model, where is the number of shares of the stock S and units of the bond B . In order to solve it for the (dynamic) portfolio of claim, by trading between stock and bond, we have:

$$\alpha S_T\left(\omega\right)+\beta=C_T\left(\omega\right) \qquad \omega=\omega_1,\omega_2$$

In order to illustrate a step in the graph of this binomial model, we can distinguish from the geometric interpretation to avoid arbitrage in this model, $C_0=\alpha S_0+\beta$ should be zero. The model shows that, since there are two equations and two unknowns in the model, the two unknowns can be easily solved.

It is easy to solve the system for one of the unknowns, either or, we have

$$\alpha = \frac{C_T(\omega_1) - C_T(\omega_2)}{S_T(\omega_1) - S_T(\omega_2)} = \frac{C_T(\omega_1) - C_T(\omega_2)}{S_0(u-d)} \tag{2.3}$$

$$\beta = \frac{uC_T(\omega_1) - dC_T(\omega_2)}{u-d} \tag{2.4}$$

According to the law of one price, we get the price of C_0 the claim; we can find the price of by recursiveness backward in time. It is the cost of the portfolio φ at time $t=0$. We put 0 instead of T in the above equations (2.3) and (2.4), we have

$$C_0 = \alpha S_0 + \beta = \frac{(u-1)C_T(\omega_1) - (1-d)C_T(\omega_2)}{u-d} \tag{2.5}$$

Formula (2.5) shows that the claim C_T can uniquely be hedged by the portfolio (α, β) , and all the risk for writing the claim C_T can be completely eliminated by the following hedging portfolio. We set the probabilities $\tilde{p}$ and $\tilde{q}$ to

$$p = \frac{1-d}{u-d} \quad p = \frac{1-d}{u-d}$$

For C_0 we can write in this form

In formula (2:6) the expectation E is risk neutral expectation, under the risk neutral probability $\tilde{p}$ and $\tilde{q}$, where it called a risk neutral valuation for a claim. We have constant rate interest in the model.

3 IN COMPLETE ONE PERIOD MODEL.

From the definition of the incomplete market, we can point out that every claim cannot be perfectly replicated by a (dynamic) portfolio. All risks cannot be eliminated from the strategy. This is the case in most real markets, based on market friction and non-trading assets.

3.1 Model Set-up.

The first phase of the binomial model in the market environment, a riskless asset and risk assets. At present, the interest rate of the model is zero, but in general, it can be added to the model. From the two risk assets in the market, one kind of asset is traded, and the second types of risk assets are non tradable, which means the incompleteness of the model. The riskless asset is B0 = BT = 1. The two risky asset S and Y are given by

$$\begin{aligned} S_T &= S_0\xi \;\; \xi = \xi^u, \xi^d \;\; 0 < \xi^d < 1 < \xi^u \\ Y_T &= Y_0\eta \;\; \eta = \eta^u, \eta^d \;\; \eta = \eta^u, \eta^d \end{aligned} \tag{3.7}$$

The randomness of assets S and Y are given by the probability space $(\Omega, \mathcal{F}_T, P)$, we have

$$\Omega = \{\omega_1\omega_2\omega_3\omega_4\} \text{ and } P(\omega_i) > 0 (1 \le i \le 4), \mathcal{F}_T = 2^\Omega$$

$$S_T(\omega_1) = S_0\xi^u, Y_T(\omega_1) = Y_0\eta^u, S_T(\omega_2) = S_0\xi^u, Y_T(\omega_2) = Y_0\eta^d,$$

$$S_T(\omega_3) = S_0\xi^d, Y_T(\omega_3) = Y_0\eta^u, S_T(\omega_4) = S_0\xi^d, Y_T(\omega_4) = Y_0\eta^d,$$

where the measure P represent the historical probability measure, in which the σ -algebra $\mathcal{F}_T$ is agree with the σ-algebra $\mathcal{F}_T^{(S,Y)}$ is generated by random variables S_T and Y_T . We can see that the σ-algebra $\mathcal{F}_T^{(S)}$ is also generated by random variables S_T .

3.2 Utility Indifference Pricing

Utility Indifference pricing method is portfolio optimization. Let us suppose that, we want to price a contingent claim with random payoff C_T , the underlying assets are a money market account and two risky assets, and we have stochastic processes on a filtered probability space $\left(\Omega, \mathcal{F}_T, P\right)$.

Let U be a Utility function measure investor preference in probability space. In general, it is assumed that the utility function U is increasing, because any investor in the market is willing to have more wealth than less wealth. And utility function U is strictly concave because the investor is risk averse. When using a utility indifference pricing, we compare two scenarios that are, "to invest in the claim" versus "do not invest in the claim". Suppose that, C_T be a claim written on S and Y in a probability space $\left(\Omega, \mathcal{F}_T, P\right)$. Due to incompleteness, in general for C_T may have different method of valuation. The definition of utility indifference pricing based on exponential utility:

$$U\left(x\right) = -e^{-\gamma x} \quad , \quad x \in \mathbb{R}$$

where $\gamma > 0$. That $U\left(x\right)$ twice continuously differentiable $U' > 0$ and $U'' < 0$ so U satisfies the definition for utility function.

Let $\varphi = \left(\alpha, \beta\right)$ be a combination includes of α shares of the traded risky asset S , and β units the riskless asset B. Its initial value $X_0 = x$ is given by $\alpha S_0 + \beta = x$, and it value at time T is given by

$$X_T = \alpha S_T + \beta = x + \alpha\left(S_T - S_0\right) \tag{3.8}$$

where X_T it is the wealth of the initial wealth . The value of the function of the claim C_T in term of exponential utility U is defined by,

$$V^{C_T}\left(X\right)\sum_{\alpha} E^P\left[U\left(X_T + C_T\right)\right] = \sup E^P\left[-e^{-\gamma\left(X_T + C_T\right)}\right] = e^{-\gamma x} \sup_{\alpha} \beta E\left[-e^{-\gamma x\left(S_T - S_0\right) + \gamma C_T}\right] \tag{3.9}$$

Here, we will define the undifferentiated price of the buyer and seller in the above definition. If one has P dollars, invest in the money market and the stock market. The utility indifference buyer price p^b is the price at which the4 investor is indifferent (in the sense that his expected utility under optimal trading is unchanged)[1] between paying nothing and note having the C_T and paying now to receive the claim p^b at time T . The price of the buyer is p^b , which is the solution to the problem

$$V\left(x - p^b\left(k\right), k\right) = V\left(x, 0\right) \tag{3.10}$$

Similarly, the utility undifferentiated seller price p^s tis the minimum amount that an investor is willing to accept for the sale of the claim C_T. That is, p^s is the solution to

$$V\left(x + p^s\left(k\right), -k\right) = V\left(x, 0\right) \tag{3.11}$$

where k is number of units of claim C_T .

Definition 3.1. The indifference price of the claim $C_T = c\left(S_T, Y_T\right)$ is defined as the amount of $\upsilon\left(C_T\right)$ for which the two value function V^{C_T} and V^0 , defined in (3.9) and corresponding, respectively, to the claim C_T and 0 coincide and for the amount $\upsilon\left(C_T\right)$ we have

$$V^0\left(x\right) = V^{C_T}\left(x + \upsilon\left(C_T\right)\right) \tag{3.12}$$

The initial wealth is x from R. In this definition, investors don't care about paying anything from his wealth without having the C_T , and paying $\upsilon\left(C_T\right)$ of his wealth to receive the claim C_T at T .

Remark 3.1. In classical no arbitrage pricing theory a complete market model gives the price of a claim C_T as $C\left(C_T\right) = E^p\left(C_T\right)$ under the assumption that the interest rate is zero. Where $\tilde{p}$ is risk neutral measure, under a unique risk neutral measure, the price is a linear function of the expected (discounted) income, called the martingale measure. However, in a incomplete model, this situation is no longer true: the claim price in a incomplete market is no longer a linear function of the expected discount revenue under a unique equivalent martingale measure. The valuation of the function is not linear, and it is an equivalent measure of incomplete market pricing. We want to look for an equivalent measure of an incomplete market model, and we look back at the formula and use zero interest rates.

$$C\left(C_T\right) = \varepsilon^Q\left(C_T\right), \tag{3.13}$$

where the ε^Q is a nonlinear functional and Q is an equivalent martingale measure. The advantage of this formula is that, the prices are expressed in one measure. However, for (3.13) to hold some regularity property should hold.

Remark 3.2. Consider two special cases:

$$\text{(a).}\quad C_T = c(S_T);$$
$$\text{(b).}\quad C_T = c(Y_T).$$

In first case (a), the randomness of asset Y which is nontraded asset in this market has no effects on the price of the claim C_T at all. Therefore, the classical no-arbitrage pricing method, which is risk neutral method applicable to the claim for pricing $\upsilon(c(C_T)) = E^p[c(S_T)]$. So the indifference pricing method is turning into no-arbitrage price.

In second case (b), we assume that the two assets (S_T and Y_T) are independent under the historical measure P so the value for $V^{C_T} = V^{c(Y_T)}$ nontrade asset is reduced to:

Recall definition (3.10), it becomes

$$V^{c(Y_T)}(x) = -e^{-\gamma x} E^P\left[e^{-\gamma c(Y_T)}\right] \sup_\alpha \mathrm{E}^P\left[-e^{-\gamma x(S_T - S_0)}\right] \tag{3.14}$$

It is straightforward from (3.14) to substitute zero instead of $c(Y_T)$, because Y_T is a nontrade asset

$$V^0(x) = -e^{-\gamma x} \sup_\alpha \mathrm{E}^P\left[-e^{-\gamma x(S_T - S_0)}\right] \tag{3.15}$$

Putting together (3.14) and (3.15) , $V^{c(Y_T)}$ becomes

$$V^{c(Y_T)}(x) = V^0(x) \mathrm{E}^P\left[e^{\gamma x(Y_T)}\right] \tag{3.16}$$

We notice that, by definition (3.8), is the solution to the following system:

$$\begin{aligned} V^0(x) &= V^{c(Y_T)}\left(x + \upsilon(c(Y_T))\right) \\ &= V^0\left(x + \upsilon(c(Y_T))\right) \mathrm{E}^P\left[e^{\gamma x(Y_T)}\right] \\ &= V^0(x) - e^{-\gamma x(c(Y_T))} \mathrm{E}^P\left[e^{\gamma x(Y_T)}\right] \\ &= -e^{-\gamma x(c(Y_T))} \mathrm{E}^P\left[e^{\gamma x(Y_T)}\right] \end{aligned}$$

Consequently, it becomes

$$\upsilon(c(Y_T)) = \frac{1}{\gamma} \log E^P\left[e^{\gamma c(Y_T)}\right] \tag{3.17}$$

From formula (3.16), we can simplify the undifferentiated price to the principle of actuarial valuation, that is, the equivalent value of certainty. When the P measure of history is used as a pricing method, the income is nonlinear.

Remark 3.3. Consider the claim C_T , decompose in the following way

$$C_T = c_1(S_T) + c_2(Y_T);$$

In this case, it may lead to wrong ideas, to price the above claim as first pricing the claim $c_1(S_T)$ by traditional no-arbitrage risk neutral valuation method, and then pricing the claim $c_2(Y_T)$ by the actuarial certainty equivalent value principle, in the final put them together as the pricing of the claim $C_T = c_1(S_T) + c_2(Y_T)$, this means,

$$\upsilon(c_1(S_T) + c_2(Y_T)) \neq C_T = \upsilon(c_1(S_T)) + \upsilon(c_2(Y_T)),$$

Even so the assets and are independent.

3.3 Nonlinear Pricing Formula

Proposition 3.1 Let Q is an equivalent martingale measure under the transaction assets S is a martingale, and at the same time, the conditional distribution of non-trading assets, given the traded one, is preserved with respect to the historical measure P , that is:

$$Q(Y_S | S_T) = P(Y_S | S_T) \tag{3.18}$$

Let $C_T = c(S_T, Y_T)$ be the claim, which is priced under exponential preference with risk aversion coefficient γ . Then the indifference price of C_T is given by

$$\upsilon(C_T)=\varepsilon^Q(C_T)=\mathbb{E}^Q\left[\frac{1}{\gamma}\log\mathbb{E}^Q\left(e^{\gamma C_T}|\mathrm{S}_T\right)\right] \tag{3.19}$$

Proof. We are going to prove the (3.19) by computing the price $\upsilon(C_T)$ from definition (3.9) and then verifying it equal to the right side of (3.19). Let

$$c_i=C_T(\omega_i)=c\left(S_T(\omega_i),Y_T(\omega_i)\right)\quad,\qquad i=1,2,3,4.$$

Recall the definition of utility indifference pricing,

$$V^{C_T}(x)=-e^{-\gamma x}\sup_\alpha\mathrm{E}^P\left[-e^{-\gamma\alpha(S_T-S_0)+\gamma C_T}\right].$$

Simply putting the value into the above formula, we get

$$V^{C_T}(x)=-e^{-\gamma x}\sup_\alpha\sum_{i=1}^{4}-p_i\left[-e^{-\gamma\alpha(S_T(\omega_i)-S_0)+\gamma C_T(\omega_i)}\right]$$
$$=-e^{-\gamma x}\sup_\alpha\left\{-e^{-\gamma\alpha S_0(\xi^u-1)}\left(p_1e^{\gamma c_1}+p_2e^{\gamma c_2}\right)-e^{-\gamma\alpha S_0(\xi^d-1)}\left(p_3e^{\gamma c_1}+p_4e^{\gamma c_2}\right)\right\}=-e^{-\gamma x}\sup_\alpha g(\alpha)$$

Taking the derivative with respect to $g(\alpha)$ and then, solving it for the $g(\alpha)$, becomes

$$\alpha^*=\frac{1}{\gamma S_0(\xi^u-\xi^d)}\log\frac{(\xi^u-1)\left(p_1e^{\gamma c_1}+p_2e^{\gamma c_2}\right)}{(1-\xi^d)\left(p_3e^{\gamma c_3}+p_4e^{\gamma c_4}\right)}.$$

By adjusting these two probabilities

$$q=\frac{1-\xi^d}{\xi^u-\xi^d}\quad,\quad 1-q=\frac{\xi^u-1}{\xi^u-\xi^d}\quad,$$

Scaling (3.20), we get

$$\alpha^*S_0\gamma(\xi^u-1)=\frac{(\xi^u-1)}{\gamma S_0(\xi^u-\xi^d)}\log\frac{(\xi^u-1)\left(p_1e^{\gamma c_1}+p_2e^{\gamma c_2}\right)}{(1-\xi^d)\left(p_3e^{\gamma c_3}+p_4e^{\gamma c_4}\right)}$$
$$\alpha^*S_0\gamma(1-\xi^d)=\frac{(1-\xi^d)}{\gamma S_0(\xi^u-\xi^d)}\log\frac{(\xi^u-1)\left(p_1e^{\gamma c_1}+p_2e^{\gamma c_2}\right)}{(1-\xi^d)\left(p_3e^{\gamma c_3}+p_4e^{\gamma c_4}\right)}.$$

Substituting α^* to $g(\alpha)$, it becomes

$$V^{C_T}(x)=-e^{-\gamma x}g(\alpha^*)$$
$$=-e^{-\gamma x}\left[-e^{-\gamma\alpha^*S_0(\xi^u-1)}\left(p_1e^{\gamma c_1}+p_2e^{\gamma c_2}\right)-e^{-\gamma\alpha^*S_0(\xi^d-1)}\left(p_3e^{\gamma c_1}+p_4e^{\gamma c_2}\right)\right]$$
$$=-e^{-\gamma x}\begin{bmatrix}-\left(p_1e^{\gamma c_1}+p_2e^{\gamma c_2}\right)\left(\dfrac{(\xi^u-1)\left(p_1e^{\gamma c_1}+p_2e^{\gamma c_2}\right)}{(1-\xi^d)\left(p_3e^{\gamma c_3}+p_4e^{\gamma c_4}\right)}\right)^{-(1-q)}\\ -\left(p_3e^{\gamma c_1}+p_4e^{\gamma c_2}\right)\left(\dfrac{(\xi^u-1)\left(p_1e^{\gamma c_1}+p_2e^{\gamma c_2}\right)^q}{(1-\xi^d)\left(p_3e^{\gamma c_3}+p_4e^{\gamma c_4}\right)}\right)\end{bmatrix}$$
$$=-e^{-\gamma x}\left(p_1e^{\gamma c_1}+p_2e^{\gamma c_2}\right)^q\left(p_1e^{\gamma c_1}+p_2e^{\gamma c_2}\right)^{(1-q)}\left[\left(\frac{q}{1-q}\right)^{1-q}\left(\frac{1-q}{q}\right)^q\right]$$

Doing some calculation turn out

$$V^{C_T}(x) = -\frac{e^{-\gamma x}\left(p_1 e^{\gamma c_1} + p_2 e^{\gamma c_2}\right)^q \left(p_1 e^{\gamma c_1} + p_2 e^{\gamma c_2}\right)^{(1-q)}}{q^q (1-q)^{1-q}} \quad . \tag{3.22}$$

Inserting values for $C_T = 0, \text{i.e.}\left(c_1 = c_2 = c_3 = c_4 = 0\right)$, we get

$$V^0(x) = -e^{-\gamma x}\left(\frac{p_1 + p_2}{q}\right)^q \left(\frac{p_3 + p_4}{q}\right)^{1-q} \tag{3.23}$$

Solve it for $\upsilon\left(C_T\right)$, we have

$$V^0(x) = V^{C_T}\left(x + \upsilon\left(C_T\right)\right) \tag{3.24}$$

Which reduces to (via (3.22) and (3.23))

$$\left(p_1 + p_2\right)^q \left(p_3 + p_4\right)^{1-q} = e^{-\gamma x(C_T)}\left(p_1 e^{\gamma c_1} + p_2 e^{\gamma c_2}\right)^q \left(p_1 e^{\gamma c_1} + p_2 e^{\gamma c_2}\right)^{1-q} \tag{3.25}$$

Consequently,

$$\upsilon\left(C_T\right) = q\frac{1}{\gamma}\log\frac{p_1 e^{\gamma c_1} + p_2 e^{\gamma c_2}}{p_1 + p_2} + \left(1-q\right)\frac{1}{\gamma}\log\frac{p_3 e^{\gamma c_3} + p_4 e^{\gamma c_4}}{p_3 + p_4} \tag{3.26}$$

We note that (3.26) the term in the log function can be expressed as a conditional expectation of $e^{-\gamma C_T}$ under the historical measure P it is follows by:

$$\frac{p_1 e^{\gamma c_1} + p_2 e^{\gamma c_2}}{p_1 + p_2} = \mathbb{E}^P\left(e^{-\gamma C_T} | A\right) \quad . \tag{3.27}$$

and

$$\frac{p_3 e^{\gamma c_3} + p_4 e^{\gamma c_4}}{p_3 + p_4} = \mathbb{E}^P\left(e^{-\gamma C_T} | A^c\right) \tag{3.28}$$

Where

$$A = \left(\omega_1, \omega_2\right) = \left\{\omega : \mathrm{S}_T(\omega) = S_T \xi^u\right\}$$
$$A^c = \left(\omega_3, \omega_4\right) = \left\{\omega : \mathrm{S}_T(\omega) = S_T \xi^d\right\}$$

We are going to seek the desired measure Q with distributions

$$Q\left(\omega_i\right) = q_i \ , \ i = 1, 2, 3, 4.$$

There is, we should designate the value q , for $i = 1, 2, 3, 4.$ In a way that $q_1 + q_2 = q$ and $q_3 + q_4 = 1 - q$, where has given in (3.21).

Consider the conditional probability $Q\left(Y_T = Y_0 9\eta^u | \mathrm{S}_T = S_0 \xi \eta^u |\right)$. From the fact in (3.18), we easily get

$$Q\left(Y_T = Y_0 9\eta^u | \mathrm{S}_T = S_0 \xi \eta^u |\right) = P\left(Y_T = Y_0 9\eta^u | \mathrm{S}_T = S_0 \xi \eta^u |\right) \quad .$$

It is reducing to

$$Q\left(\omega_1\omega_2 | \omega_3\omega_4\right) = P\left(\omega_3\omega_4 | \omega_1\omega_2\right) \Rightarrow \frac{q_1}{q_1 + q_2} = \frac{p_1}{p_1 + p_2} \quad .$$

In the same way, we have other numbers,

$$\frac{q_2}{q_1 + q_2} = \frac{p_2}{p_1 + p_2}; \frac{q_3}{q_3 + q_4} = \frac{p_3}{p_3 + p_4}; \frac{q_4}{q_3 + q_4} = \frac{p_4}{p_3 + p_4} \quad ,$$

Repeat the same idea in different form by (nothing $q = q_1 + q_2$)

$$q_i = q\frac{p_i}{p_1+p_2}(i=1,2) \text{ and for } q_i = (1-q)\frac{p_i}{p_3+p_4}(i=3,4) \tag{3.29}$$

Next, we have

$$\log \mathbb{E}^Q\left(e^{\gamma C_T|S_T}\right) = \left(\log \mathbb{E}^Q\left(e^{\gamma C_T|S_T}\right)\right)\mathbb{I}_A + \left(\log \mathbb{E}^Q\left(e^{\gamma C_T|S_T}\right)\right)\mathbb{I}_{A^c}$$

$$\log \mathbb{E}^P\left(e^{\gamma C_T|S_T}\right) = \left(\log \mathbb{E}^P\left(e^{\gamma C_T|S_T}\right)\right)\mathbb{I}_A + \left(\log \mathbb{E}^P\left(e^{\gamma C_T|S_T}\right)\right)\mathbb{I}_{A^c}$$

$$= \left(\log\frac{p_1e^{\gamma c_1}+p_2e^{\gamma c_2}}{p_1+p_2}\right)\mathbb{I}_A + \left(\log\frac{p_3e^{\gamma c_3}+p_4e^{\gamma c_4}}{p_3+p_4}\right)\mathbb{I}_{A^c} = \frac{1}{\gamma}\log \mathbb{E}^P\left(e^{\gamma C_T|S_T}\right).$$

Taking the expectation with respect to measure, we get

$$\mathbb{E}^Q\left(\frac{1}{\gamma}\log \mathbb{E}^P\left(e^{\gamma C_T|S_T}\right)\right)$$

$$= \mathbb{E}^Q\left(\frac{1}{\gamma}\left(\left(\log\frac{p_1e^{\gamma c_1}+p_2e^{\gamma c_2}}{p_1+p_2}\right)\mathbb{I}_A + \left(\log\frac{p_3e^{\gamma c_3}+p_4e^{\gamma c_4}}{p_3+p_4}\right)\mathbb{I}_{A^c}\right)\right)$$

$$= \frac{1}{\gamma}\log\frac{p_1e^{\gamma c_1}+p_2e^{\gamma c_2}}{p_1+p_2}Q(A) + \left(\frac{1}{\gamma}\log\frac{p_3e^{\gamma c_3}+p_4e^{\gamma c_4}}{p_3+p_4}\right)Q(A^c)$$

$$= \frac{1}{\gamma}\left(\log\frac{p_1e^{\gamma c_1}+p_2e^{\gamma c_2}}{p_1+p_2}\right)q + \left(\frac{1}{\gamma}\log\frac{p_3e^{\gamma c_3}+p_4e^{\gamma c_4}}{p_3+p_4}\right)(1-q) = \upsilon(C_T)$$

Therefore, proposition 3.1 is proved.

3.4 Valuation Procedure for Indifference Pricing

From formula (3.18), we can point out that, whether martingale or historical measure, under the single pricing method, the two step nonlinear process can be evaluated without distinction.

The first step is to push the risk preference into the model, distort the yield of the original derivative products to the risk preference adjustment income, which is called conditional certainty equivalence:

$$C_T = \frac{1}{\gamma}\log \mathbb{E}^P\left(e^{\gamma C_T|S_T}\right) \tag{3.30}$$

The new benefits of the claim C_T , it has an actuarial type of payoff, and it also carry on risk aversion that is based on utility methodology. But the certainty equivalence does not apply to the model. In short, in fact, we do not consider the functional of the actuarial type, since

$$C_T \neq \frac{1}{\gamma}\log \mathbb{E}^P\left(e^{\gamma C_T|S_T}\right) \text{and } C_T \neq \frac{1}{\gamma}\log \mathbb{E}^Q\left(e^{\gamma C_T|S_T}\right) \tag{3.31}$$

The second step of the evaluation is the classic non arbitrage pricing: to price the preference adjusted payoff CT, it is dependent only on a traded asset ST, it should be a non-arbitrage price. In these two steps, the same measure should be used. For a given price, we have

$$\upsilon\left(C^T\right) = \varepsilon^Q\left(C_T\right) = \mathbb{E}^Q\left(C^T\right)$$

Remember, these two steps are not completely different. The first step is nonlinear, but the second step is linear and opposite. In the pricing, a pricing measure is used throughout the work. Its basic function is that it should not be exchanged with the distribution of the conditions of the risk. It can be exchanged with our past historical values.

3.5 Properties of Indifference Prices

We differ from the previous analysis, that is, nonlinear pricing.

$$\upsilon\left(C_T\right) = \mathbb{E}^Q\left(C_T\right) - \varepsilon^Q\left(C^T\right)$$

Where the preference adjusted payoff C_T is the conditional certainty equivalent evaluation for

$$C_T = \frac{1}{\gamma}\log \mathbb{E}^P\left(e^{\gamma C_T|\mathrm{S}_T}\right)$$

and the conditional for the Q measures are: (3.18). There is a direct relationship between the pricing formula and the linear pricing formula. This is reflected in our early comprehensive model discussion.

Here, we will discuss some of the more important structural features in order to better understand the undifferentiated pricing. We write the independent of price $\upsilon(C_T)$ relate to the risk aversion coefficient , by writing it in way: $\upsilon(C_T)=\upsilon(C_T,\gamma)$.

Proposition 3.2. The price $\upsilon(C_T,\gamma)$ is an increasing and continuous function of $\gamma\in(0,\infty)$ and moreover, for claim C_T , there holds

$$\upsilon(C_T,\gamma)=\upsilon(C_T,1), \text{then } \gamma=1 .$$

Proof. Page 17

Here, we will discuss some of the more important structural features in order to better understand the undifferentiated pricing. We're writing an independent price $\upsilon(C_T)$ relate to the risk aversion coefficient γ, by writing it in way: $\upsilon(C_T)=\upsilon(C_T,\gamma)$.

Proposition 3.2 The price $\upsilon(C_T,\gamma)$ is an increasing and continuous function of $\gamma\in(0,\infty)$. And moreover, for all claim C_T, there holds

$$\upsilon(C_T,\gamma)=\upsilon(C_T,1) \text{ , then } \gamma=1. \tag{3.21}$$

Proof: Recall the formula

$$\upsilon(\mathrm{C}_T,\gamma)=\mathbb{E}^Q\left[\frac{1}{\gamma}\mathbb{E}^Q\left[\mathrm{e}^{\gamma C_T}|\mathrm{S}_T\right]\right].$$

The continuity of $\upsilon(C_T,\gamma)$ in $\gamma\in(0,\infty)$, it easily follows from the properties of conditional expectation. To prove monotonicity, let $0<\gamma_1<\gamma_2$. Then applying the Holder's inequality,

$$\left(\mathbb{E}^Q\left[e^{\gamma_1 C_T}|\mathrm{S}_T\right]\right)^{\frac{1}{\gamma_1}} \le \left(\mathbb{E}^Q\left[e^{\gamma_2 C_T}|\mathrm{S}_T\right]\right)^{\frac{1}{\gamma_2}}$$

After some calculation, we get

$$\left(\mathbb{E}^Q\left[e^{\gamma_1 C_T}|\mathrm{S}_T\right]\right) \le \left(\mathbb{E}^Q\left[e^{\gamma_2 C_T}|\mathrm{S}_T\right]\right)^{\frac{\gamma_1}{\gamma_2}}$$

And

$$\frac{1}{\gamma_1}\log\mathbb{E}^Q\left[e^{\gamma_1 C_T}|\mathrm{S}_T\right] \le \frac{1}{\gamma_2}\log\mathbb{E}^Q\left[e^{\gamma_2 C_T}|\mathrm{S}_T\right] .$$

Taking expectation with respect to the Q measure, it yields that $\upsilon(C_T,\gamma_1)\le\upsilon(C_T,\gamma_2)$.

To go through the second part of proposition, recall (3.18);

$$\upsilon(C_T,\gamma)=q\frac{1}{\gamma}\log\frac{p_1e^{\gamma c_1}+p_2e^{\gamma c_2}}{p_1+p_2}+(1-q)\frac{1}{\gamma}\log\frac{p_3e^{\gamma c_3}+p_4e^{\gamma c_4}}{p_3+p_4}$$

Substituting $\gamma=1$ instead of γ

$$\upsilon(C_T,1)=q\log\frac{p_1e^{\gamma c_1}+p_2e^{\gamma c_2}}{p_1+p_2}+(1-q)\log\frac{p_3e^{\gamma c_3}+p_4e^{\gamma c_4}}{p_3+p_4}.$$

We take a particular claim, that $c_1=C_T(\omega_1)>0$ and $c_i=C_T(\omega_i)=0$ for $i=2,3,4,$ for which all $c_1>0$, it becomes

$$\frac{1}{\gamma}\log\frac{p_1e^{\gamma c_1}+p_2}{p_1+p_2}=\log\frac{p_1e^{c_1}+p_2}{p_1+p_2},$$

differentiating with respect to c_1 , we have

$$\frac{e^{\gamma c_1}}{p_1 e^{\gamma c_1} + p_2} = \frac{e^{c_1}}{p_1 e^{c_1} + p_2} \quad ,$$

It turns into

$$e^{c_1}\left(e^{\gamma c_1} + p_2\right) = e^{c_1}\left(e^{c_1} + p_2\right) \Rightarrow e^{\gamma c_1} = e^{c_1} \quad .$$

Therefore $\gamma = 1$.

Proposition 3.3. It holds the following limiting behaviors for the claim:

$$\lim_{\gamma\to 0^+} \upsilon\left(C_T, \gamma\right) = \mathbb{E}^Q\left[C_T\right] \quad , \tag{3.25}$$

$$\lim_{\gamma\to\infty} \upsilon\left(C_T, \gamma\right) = \mathbb{E}^Q\left[\left\|C_T\right\|_{L_Q^\infty\{\cdot|C_T\}}\right]. \tag{3.26}$$

Proof. To prove (3.25), we recall the nonlinear pricing formula; namely,

$$\upsilon\left(C_T, \gamma\right) = q\frac{1}{\gamma}\log\frac{p_1 e^{\gamma c_1} + p_2 e^{\gamma c_2}}{p_1 + p_2} + \left(1-q\right)\frac{1}{\gamma}\log\frac{p_3 e^{\gamma c_3} + p_4 e^{\gamma c_4}}{p_3 + p_4} \quad .$$

Sending $\gamma \to 0^+$, and using the fact

$$q_i = q\frac{p_i}{p_1 + p_2}\left(i = 1,2\right) \text{ and for } q_i = \left(1-q\right)\frac{p_i}{p_3 + p_4}\left(i = 3,4\right)$$

$$\lim_{\gamma\to 0^+} \upsilon\left(C_T, \gamma\right) = \left[q\left(\frac{p_1 c_1}{p_1 + p_2} + \frac{c_2 p_2}{p_1 + p_2}\right) + \left(1-q\right)\left(\frac{p_3 c_3}{p_3 + p_4} + \frac{c_4 p_4}{p_3 + p_4}\right)\right]$$

we write it in more simple form, thus we have

$$\lim_{\gamma\to 0^+} \upsilon\left(C_T, \gamma\right) = \left[q\log\left(\frac{p_1 c_1 + c_2 p_2}{p_1 + p_2}\right) + \left(1-q\right)\log\left(\frac{p_3 c_3 + c_4 p_4}{p_3 + p_4}\right)\right] \quad ,$$

$$\lim_{\gamma\to 0^+} \upsilon\left(C_T, \gamma\right) = \sum_{i=1}^{4} q_i c_i = \mathbb{E}^Q\left[C_T\right] \quad .$$

To prove (3.26), recall formula (3.18),

$$\upsilon\left(C_T, \gamma\right) = q\frac{1}{\gamma}\log\frac{p_1 e^{\gamma c_1} + p_2 e^{\gamma c_2}}{p_1 + p_2} + \left(1-q\right)\frac{1}{\gamma}\log\left(\frac{p_3 e^{\gamma c_3} + p_4 e^{\gamma c_4}}{p_3 + p_4}\right).$$

Taking limit as $\gamma \to \infty$, we have

$$\upsilon\left(C_T, \gamma\right) = q\frac{1}{\gamma}\log\frac{p_1 e^{\gamma c_1} + p_1 e^{\gamma c_1}}{p_1 + p_2} + \left(1-q\right)\frac{1}{\gamma}\log\frac{p_3 e^{\gamma c_3} + p_4 e^{\gamma c_4}}{p_3 + p_4}$$

$$\lim_{\gamma\to\infty} \upsilon\left(C_T, \gamma\right) = q\max\left\{c_1, c_2\right\} + \left(1-q\right)\max\left\{c_3, c_4\right\}$$

$$\lim_{\gamma\to\infty} \upsilon\left(C_T, \gamma\right) = \mathbb{E}^Q\left[\left\|C_T\right\|_{L_Q^\infty\{\cdot|C_T\}}\right].$$

Proposition 3.4 The undifferentiated price is in accordance with the principle of no arbitrage, that is, for ry $\gamma > 0$,

$$\inf_{Q\in Q_e} \mathbb{E}^Q\left[C_T\right] \le \upsilon\left(C_T, \gamma\right) \le \sup_{Q\in Q_e} \mathbb{E}^Q\left[C_T\right] \quad . \tag{3.27}$$

Here Q_e is the set of equivalent martingales measure which are equivalent to P .

To prove the monotony, it is always necessary to have $c_1 < c_2$ and $c_3 < c_4$ with respect to the risk aversion, applying the proposition 3.3,

$$\mathbb{E}^Q\left[C_T\right] \le \upsilon\left(C_T, \gamma\right) \le \mathbb{E}^Q\left[\left\|C_T\right\|_{L_Q^\infty\{\cdot|C_T\}}\right] \quad .$$

Taking infimum over the set of (3.27), it is suffices to observe the necessary condition, we assume that, $c_1 < c_2$ and $c_3 < c_4$

$$\mathbb{E}^Q\left[\left\|C_T\right\|_{L_Q^\infty\{\cdot|C_T\}}\right] = q\max\{c_1, c_2\} + (1-q)\max\{c_3, c_4\} = \mathbb{E}^{\bar{Q}}[C_T],$$

Where $\overline{Q}$ is also martingale measure with some elementary probabilities, we have,

$$\overline{Q}(\omega_1) = 0, \overline{Q}(\omega_2) = q, \overline{Q}(\omega_3) = 0, \overline{Q}(\omega_4) = 1-q \quad .$$

Then

$$\upsilon(C_T, \gamma) \le \mathbb{E}^Q\left[\left\|C_T\right\|_{L_Q^\infty\{\cdot|C_T\}}\right] = \mathbb{E}^{\bar{Q}}[C_T] \le \sup_{Q\in Q_E}\mathbb{E}^Q[C_T].$$

Proposition 3.5 Indifference price $\upsilon(C_T)$ is increasing and convex function of payoff C_T , which the following are satisfied,

$$\text{I. If } C\frac{1}{T} \le C\frac{2}{T} \text{ , then } \upsilon(C_T^1) \le \upsilon(C_T^2) \quad , \tag{3.28}$$

$$\text{II. For every } \propto \in (0,1), \upsilon\left(\propto C_T^1 + (1-\propto)C_T^2\right) \le \propto \upsilon(C_T^1) + (1-\propto)\upsilon C_T^2. \tag{3.28}$$

Proof. Recall the formula

$$\upsilon(C_T) = \mathbb{E}^Q\left[\frac{1}{\gamma}\log\mathbb{E}^Q\left[e^{\gamma C_T}|S_T\right]\right] \quad .$$

To prove (3.28), since $\upsilon(C_T)$ is increasing in $C_T^1 \le C_T^2$, putting these values into the above equation, we get

$$\upsilon(C_T^1) = \mathbb{E}^Q\left[\frac{1}{\gamma}\log\mathbb{E}^Q\left[e^{\gamma C_T^1}|S_T\right]\right], \upsilon(C_T^2) = \mathbb{E}^Q\left[\frac{1}{\gamma}\log\mathbb{E}^Q\left[e^{\gamma C_T^2}|S_T\right]\right] \quad ,$$

From the fact $C_T^1 \le C_T^2$, then we can write them in term of above pricing formula,

$$\mathbb{E}^Q\left[\frac{1}{\gamma}\log\mathbb{E}^Q\left[e^{\gamma C_T^1}|S_T\right]\right] \le \mathbb{E}^Q\left[\frac{1}{\gamma}\log\mathbb{E}^Q\left[e^{\gamma C_T^2}|S_T\right]\right] \Rightarrow \upsilon(C_T^1) \le \upsilon(C_T^2) \quad .$$

To show the (3.29), applying Holder's Inequality, we get,

$$\upsilon\left(\propto C_T^1 + (1-\propto)C_T^2\right) \le \propto \upsilon(C_T^1) + (1-\propto)\upsilon C_T^2$$

by definition of conditional certainty equivalent,

$$= \mathbb{E}^Q\left[\frac{1}{\gamma}\log\mathbb{E}^Q\left[e^{\gamma\left(\propto C_T^1 + (1-\propto)C_T^2\right)}|S_T\right]\right]$$

$$\le \mathbb{E}^Q\left[\frac{1}{\gamma}\log\left(\left(\mathbb{E}^Q\left[e^{\gamma C_T^1}|S_T\right]\right)^{\propto}\left(\mathbb{E}^Q\left[e^{\gamma C_T^2}|S_T\right]\right)^{(1-\propto)}\right)\right]$$

$$= \propto\left[\frac{1}{\gamma}\log\left(\mathbb{E}^Q\left[e^{\gamma C_T^1}|S_T\right]\right)\right] + (1-\propto)\mathbb{E}^Q\left[\frac{1}{\gamma}\log\left(\mathbb{E}^Q\left[e^{\gamma C_T^2}|S_T\right]\right)\right]$$

$$\le \propto \upsilon(C_T^1) + (1-\propto)\upsilon(C_T^2)$$

Hence convexity is proved.

Proposition 3.6 The indifference price of $\upsilon(C_T)$ satisfies the following properties:

$$\upsilon(\propto C_T) \le \propto \upsilon(C_T) \textit{ for every } \propto \in (0,1) \quad , \tag{3.30}$$

and

$$\upsilon(\propto C_T) \le \propto \upsilon(C_T) \textit{ for every } \propto \ge 1 \tag{3.31}$$

Proof. From proposition 3.5, we have

$$\upsilon(\propto \mathrm{C}_T) = \mathbb{E}^Q\left[\frac{1}{\gamma}\log\left(\mathbb{E}^Q\left[e^{\gamma\propto C_T}|S_T\right]\right)\right] .$$

We observe that, if $\propto \in (0,1)$, then setting the $\bar{\gamma} = \propto \gamma < \gamma$, it becomes

$$\upsilon(\propto C_T) = \upsilon(\propto C_T, \gamma) = \propto \mathbb{E}^Q\left[\frac{1}{\gamma}\log\left(\mathbb{E}^Q\left[e^{\gamma C_T}|S_T\right]\right)\right] = \propto \upsilon\left(C_T, \bar{\gamma}\right) .$$

Since $\upsilon(\propto C_T, \gamma)$ is increasing function in γ , we can derive the relation from (3.30), that is

$$\upsilon(\propto C_T) = \propto \upsilon\left(C_T, \bar{\gamma}\right) \leq \propto \upsilon(C_T, \gamma) = \propto \upsilon(C_T).$$

To prove (3.31), using the monotonicity of price with respect to the risk aversion. If $\propto > 1$ then $\bar{\gamma} > \gamma$.

Hence form the increasingness of $\upsilon(\propto C_T, \gamma)$ we get,

$$\upsilon(\propto C_T) = \propto \upsilon\left(C_T, \bar{\gamma}\right) \geq \propto \upsilon(C_T, \gamma) = \propto \upsilon(C_T) .$$

4 CONVEX RISK MEASURE

In simple terms, there are two possible explanations for a risk measure: they are considered a pricing rule and a capital requirement rule. The convex risk measure has a certain nature and is very important in the C_T meaning of the claim's income. There is a very close relationship with the undifferentiated pricing.

Definition 4.1 A mapping form $\varphi : F_T \to R$ is said to be convex risk measure, if it satisfies the following properties, for every $C_T, C_T^1, C_T^2 \in F_T$.

Ⅰ. Convexity: $\forall \alpha[0,1], \varphi\left(\propto C_T^1 + (1-\propto)C_T^2\right) \leq \propto \varphi\left(C_T^1\right) + (1-\propto)\varphi\left(C_T^2\right)$.

Ⅱ. Monotonicity: if $\left(C_T^1\right) \leq \left(C_T^2\right)$, then $\varphi\left(C_T^1\right) \geq \varphi\left(C_T^2\right)$.

Ⅲ.Translation invariance : for $\forall q \in R$, then $\varphi(C_T + q) = \varphi(C_T) - q$.

To define a mapping on $C_T \in F_T$ by

$$\varphi(C_T) = \upsilon(-C_T) = \mathbb{E}^Q\left[\frac{1}{\gamma}\log\left(\mathbb{E}^Q\left[e^{-\gamma C_T}|S_T\right]\right)\right] \tag{4.1}$$

Note: The number $\upsilon(C_T)$ in the indifference pricing is the value of the pay off C_T , whereas the $\varphi(C_T) = \upsilon(-C_T)$ interpreted as a capital requirement imposed by supervising body or company to accept the position C_T .

Proposition 4.1 In (4.1) the mapping which is given, it called a convex risk measure.

$$\text{Ⅰ. Convexity:} \quad \forall \alpha[0,1], \varphi\left(\propto C_T^1 + (1-\propto)C_T^2\right) \leq \propto \varphi\left(C_T^1\right) + (1-\propto)\varphi\left(C_T^2\right) \tag{4.2}$$

Proof. For the convexity, we applying Holder's inequality, it becomes,

$$\varphi\left(\propto\left(C_T^1\right) + (1-\propto)\left(C_T^2\right)\right) = \upsilon\left(-\left(\propto\left(C_T^1\right) + (1-\propto)\left(C_T^2\right)\right)\right)$$

$$= \mathbb{E}^Q\left[\frac{1}{\gamma}\log \mathbb{E}^Q\left[e^{\gamma\left(-\propto C_T^1 + (1-\propto)\left(C_T^2\right)\right)}|\mathrm{S}_T\right]\right] ,$$

By Holder's inequality $\leq \left[\mathbb{E}^Q\left(\frac{1}{\gamma}\log \mathbb{E}^Q\left[e^{-\gamma C_T^1}|\mathrm{S}_T\right]\right)^{\propto}\left(\mathbb{E}^Q\left[e^{-\gamma C_T^2}|\mathrm{S}_T\right]\right)^{(1-\propto)}\right]$

$$= \propto \mathbb{E}^Q\left[\frac{1}{\gamma}\log\left(\mathbb{E}^Q\left[e^{-\gamma C_T^1}|\mathrm{S}_T\right]\right)\right] + (1-\propto)\mathbb{E}^Q\left[\frac{1}{\gamma}\log\left(\mathbb{E}^Q\left[e^{-\gamma C_T^1}|\mathrm{S}_T\right]\right)\right].$$

$$\leq \propto \varphi\left(C_T^1\right) + (1-\propto)\varphi\left(C_T^2\right)$$

Ⅱ. Monotonicity: if $\left(C_T^1\right) \le \left(C_T^2\right)$, then $\varphi\left(C_T^1\right) \ge \varphi\left(C_T^2\right)$. (4.3)

Proof: To prove monotonicity; since $C_T^1 \le C_T^2$, the risk measurement should be reduced by the value added value. If investors are more willing to get more returns from their investment, we can change it to financial terms, and the return on investment is low.

$$\upsilon\left(C_T^1\right) = \mathbb{E}^Q\left[\frac{1}{\gamma}\log \mathbb{E}^Q\left[e^{\gamma C_T^1}|\mathrm{S}_T\right]\right], \upsilon\left(C_T^2\right) = \mathbb{E}^Q\left[\frac{1}{\gamma}\log \mathbb{E}^Q\left[e^{\gamma C_T^2}|\mathrm{S}_T\right]\right]$$

$$\varphi\left(C_T^1\right) = \upsilon\left(-C_T^1\right) = \mathbb{E}^Q\left[\frac{1}{\gamma}\log \mathbb{E}^Q\left[e^{-\gamma C_T^1}|\mathrm{S}_T\right]\right], \varphi\left(C_T^2\right) = \upsilon\left(-C_T^2\right) = \mathbb{E}^Q\left[\frac{1}{\gamma}\log \mathbb{E}^Q\left[e^{-\gamma C_T^2}|\mathrm{S}_T\right]\right]$$

becomes

$$C_T^1 \le C_T^2, \upsilon\left(C_T^1\right) \le \upsilon\left(C_T^2\right) \Rightarrow \varphi\left(C_T^1\right) \ge \varphi\left(C_T^2\right),$$

Ⅲ.Translation invariance : for $\forall q \in R$, then $\varphi\left(C_T + q\right) = \varphi\left(C_T\right) - q$. (4.4)

Proof: To prove translation invariance, we recall definition (4.1). Since $q \in R$, and $\varphi\left(C_T + q\right)$, setting the values into (4.1), it becomes

$$\begin{aligned}
\varphi\left(C_T + q\right) = \upsilon\left(-\left(C_T + q\right)\right) &= \mathbb{E}^Q\left[\frac{1}{\gamma}\log\left(\mathbb{E}^Q\left[e^{\gamma\left(-\left(C_T+q\right)\right)}|\mathrm{S}_T\right]\right)\right] \\
&= \mathbb{E}^Q\left[\frac{1}{\gamma}\log\left(\mathbb{E}^Q\left[e^{-\gamma C_T^1 - \gamma q}|\mathrm{S}_T\right]\right)\right] \\
&= \mathbb{E}^Q\left[\frac{1}{\gamma}\log\left(\left(\mathbb{E}^Q\left[e^{-\gamma C_T}|\mathrm{S}_T\right]\right)\left(\mathbb{E}^Q\left[e^{-\gamma q}\right]\right)\right)\right] \\
&= \mathbb{E}^Q\left[\frac{1}{\gamma}\log\left(\mathbb{E}^Q\left[e^{-\gamma C_T}|\mathrm{S}_T\right]\right) + \frac{1}{\gamma}\left(-\gamma q\right)\right] \\
&= \mathbb{E}^Q\left[\frac{1}{\gamma}\log\left(\mathbb{E}^Q\left[e^{-\gamma C_T}|\mathrm{S}_T\right]\right) - \mathbb{E}^Q\left[q\right]\right] \\
&= \mathbb{E}^Q\left[\frac{1}{\gamma}\log\left(\mathbb{E}^Q\left[e^{\left(-\gamma C_T\right)}|\mathrm{S}_T\right]\right)\right] - q \\
&= \varphi\left(C_T\right) - q
\end{aligned}$$

Remark 4.1 In convex risk measurement, convexity means that diversification should not be increased. In order to understand better, any convex combination that allows the risk must be admissible.

Remark 4.2 In financial mathematics, monotonicity refers to the reduction in the risk of the lower side when the income increases.

Remark 4.3 The interpretation of translation invariance is also a very important nature of the convex risk measurement. $\varphi\left(C_T\right)$ may be interpreted as the amount the agent has to hold to completely cancel risk associated with his risky position in claim $\varphi\left(C_T\right)$.

Proposition 4.2 An agent, whom wants to completely cancel risk from the risky position $\varphi\left(C_T\right)$, given by

$$\varphi\left(C_T + \varphi\left(C_T\right)\right) = \varphi\left(C_T\right) - \varphi\left(C_T\right) = 0 \qquad (4.5)$$

Proof. To conclude the (4.5), recall definition (4.1),

$$\varphi\left(C_T + \varphi\left(C_T\right)\right) = \varphi\left(C_T\right) + \varphi\left(\varphi\left(C_T\right)\right)$$

$$\varphi\left(C_T\right) = \upsilon\left(-C_T\right) = \mathbb{E}^Q\left[\frac{1}{\gamma}\log\left(\mathbb{E}^Q\left[e^{-\gamma C_T}|\mathrm{S}_T\right]\right)\right]$$

$$\varphi(\varphi(C_T)) = \upsilon(-(-C_T)) = -\upsilon(-C_T) = -\mathbb{E}^Q\left[\frac{1}{\gamma}\log\left(\mathbb{E}^Q\left[e^{-\gamma C_T}|S_T\right]\right)\right]$$
$$= \mathbb{E}^Q\left[\frac{1}{\gamma}\log\left(\mathbb{E}^Q\left[e^{-\gamma C_T}|S_T\right]\right)\right] - \mathbb{E}^Q\left[\frac{1}{\gamma}\log\left(\mathbb{E}^Q\left[e^{-\gamma C_T}|S_T\right]\right)\right]$$

From the fact that $\varphi(C_T) + (-\varphi(C_T))$,

$$= \varphi(C_T) - \varphi(C_T)$$
$$= \varphi(C_T) - \varphi(C_T) = 0 .$$

5 EXPECTED UTILITY AND INDIFFERENCE PRICING

5.1 Expected Utility and Lotteries

Modern mathematics financial theory P and P economic theory believe that if an agent faces uncertainty, then their decision should be based on expected utility. That is to say, the possibility of integrating (or summarizing) the utility of wealth exceeds the result. If you remember that the utility function is a random variable, and the expectation of a random variable is a number, then it is natural.

5.2 Lotteries

Let $(\Omega, F_T, \mathbb{P})$, be a probability space. Observe that given two lotteries and, any convex combination of them: $\propto p + (1-\propto)$ P with $\propto \in [0,1]$ is also a lottery. This can be viewed simply as stating the mathematical fact that $\mathbb{P}$ is convex. We can also view $\propto p + (1-\propto)$ P more explicitly as a compound lottery, summarizing the overall probability from two successive events: first, a coin flips with weight $\propto$, $(1-\propto)$ that determines whether the lottery P or P should be used to determine the ultimate consequences; second either the lottery P or P .

5.3 Expected Utility Definition

Definition 5.1 A utility function $U : \mathbb{P} \to \mathrm{R}$ has an expected utility form (or is a von Neumann-Morgenstern utility function) if there are numbers $(u_1, \cdots, u_n)$ for each of N outcomes $(x_1, \cdots, x_n)$ such that for every $P \in \mathbb{P}$, where $(\Omega, F_T, \mathbb{P})$ is probability space, and we have $U(P) = \sum_{i=1}^{n} P_i u_i$. In case of two lotteries it will reduce to the form $\propto p + (1-\propto)$ P .Investor preferences can be expressed as expected utility functions. For example, if we have two random results, the probability space is $(\Omega, F_T, \mathbb{P})$. In order to make a preference between the two, we need to know the expected utility of two random results. To a certain extent；

$$p_1 \succ p_2 \Rightarrow \mathbb{E}[u(p_1)] \succ \mathbb{E}[u(p_2)] \tag{5.1}$$

Crucially, an expected utility function is linear in the probabilities, meaning that:

$$U(\propto p + (1-\propto)'P) = \propto U(p) + (1-\propto)U'P \ . \tag{5.2}$$

5.4 Certainty Equivalent and Expected Utility

Now let's briefly review our periodic complete model, which consists of two assets, one is a money market account and the other is a risky asset. But here, the interest rate of riskless assets is non zero. The randomness of asset S is given by a probability space $(\Omega, F_T, \mathbb{P})$, with the probability p and $(1-p)$, the probabilities are from flips of a con, but not fair coin. To evaluate the assets, from (2.5), we can write, just by adding nonzero interest rate to model,

$$S_0 = \frac{\mathbb{E}[S_1]}{1+r} = \frac{[pS_u + (1-p)S_d]}{1+r} \ . \tag{5.3}$$

Formula (5.3) is a reasonable method for pricing assets in a complete market, called a binomial non arbitrage pricing method for discrete time assets. Another pricing method, called undifferentiated pricing, is based on an exponential utility function, if one has two random results, L_1 and L_2 . In order to consider the expected utility between the two, we can write their preferences in written form,

$$L_1 \preceq L2 \Rightarrow \mathbb{E}\left[u\left(L_1\right)\right] \preceq \mathbb{E}\left[u\left(L_2\right)\right]. \tag{5.4}$$

Let L_1 and L_2 be two possible opportunities (situations) for investment in the market, and Let $X_0 = x$ be an initial wealth, to allocate the initial wealth $X_0 = x$ between, L_1 and L_2, we need to know L_1 and L_2 behaviors in the market. They are the following,

a)=(opportunity L_1).(do nothing): We have initial wealth X yuan into a savings account and increase at risk-free interest rate. Ex0pected's practical random outcomes is L_1 is just the expected utility of our initial wealth. $X_0 = x$ is current wealth. We have two possible random outcomes from this strategy. They are,

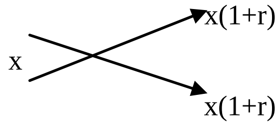

FIG 5.1 A ONE STEP BINOMIAL MODEL FOR A SAVING ACCOUNT OF INITIAL WEALTH X.

B) =(opportunity L_2),(buy an asset S): We buy the asset S, our current wealth is $\left(x - S_0\right)$, but in case, we have one unit of asset S . The two possible random results of the second strategies are,

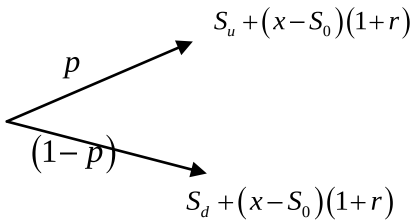

FIG 5.2 A ONE STEP BINOMIAL MODEL FOR BUY THE ASSET S

Definition 5.1 Investors are indifferent to "putting X dollars into the money market account" and "buying a share of S and liquidation." We can write it in a form.

$$\mathbb{E}\left[u\left(L_1\right)\right] = \mathbb{E}\left[u\left(L_2\right)\right] \tag{5.5}$$

In mathematical finance, the most useful utility function is exponential utility. We choose

$$u\left(x\right) = \frac{1 - e^{-\gamma x}}{\gamma} = -\frac{1}{\gamma} e^{-\gamma x}$$

Where in (5.6), γ is the risk aversion level of an agent and $X_0 = x$ is an initial wealth. If we send $x \to 0$ the , $u\left(x\right) = 0$ and by sending $x \leftrightarrow \infty$, then $u\left(x\right) = \frac{1}{\gamma}$ in the same manner we can do it for γ .

Recall (5.5)

$$\mathbb{E}\left[u\left(L_1\right)\right] = \mathbb{E}\left[u\left(L_2\right)\right] .$$

Note: $\mathbb{E}\left[u\left(L_1\right)\right]$ and $\mathbb{E}\left[u\left(L_2\right)\right]$ are the expected utilities of L_1 and L_2.

The expected utility of L_1 , is $\mathbb{E}\left[u\left(L_1\right)\right] = -\frac{1}{\gamma} e^{-\gamma x\left(1+r\right)}$, and also the expected utility of L_2, is

$$\mathbb{E}\left[u\left(L_2\right)\right] = -\frac{1}{\gamma}\left[pe^{-\gamma\left[S_u + \left(x - S_0\right)\left(1+r\right)\right]} + \left(1-p\right)e^{-\gamma\left[S_d + \left(x - S_0\right)\left(1+r\right)\right]}\right]$$

Putting the expected utilities of L_1 and L_2, together into (5.5), by definition we get

$$-\frac{1}{\gamma} e^{-\gamma x\left(r+1\right)} = -\frac{1}{\gamma}\left[pe^{-\gamma\left[S_u + \left(x - S_0\right)\left(1+r\right)\right]} + \left(1-p\right)e^{-\gamma\left[S_d + \left(x - S_0\right)\left(1+r\right)\right]}\right]$$

Where the initial wealth factors out,

$$-\frac{1}{\gamma}e^{-\gamma x(r+1)}=-\frac{1}{\gamma}e^{-\gamma x(r+1)}\left[pe^{-\gamma S_u}e^{\gamma S_0(x+1)}+(1-p)e^{-\gamma S_d}e^{\gamma S_0(x+1)}\right],$$

and

$$1=e^{\gamma S_0(r+1)}\left[pe^{-\gamma\left[S_u+(x-S_0)(1+r)\right]}+(1-p)e^{-\gamma\left[S_d+(x-S_0)(1+r)\right]}\right],$$

$$\log\left(e^{-\gamma S_0(r+1)}\right)=\log\left[pe^{-\gamma S_u}+(1-p)e^{-\gamma S_d}\right]$$

Consequently, we get

$$S_0=-\frac{1}{\gamma(1+r)}\log\left[pe^{-\gamma S_u}+(1-p)e^{-\gamma S_d}\right] \tag{5.6}$$

The calculation formula (5.6) is equivalent to the actuarial point of view for the determination of the asset price of S. Here recall that point 3.3 here, and we'll apply it to our strategy (5.6). Suppose that the investor is an infinite risk aversion. How much should he pay for the asset S ? In financial mathematics, we can answer this question by sending $\gamma\to\infty$ (= infinitely risk averse).

We re-invoke (5.6),

$$S_0=-\frac{1}{\gamma(1+r)}\log\left[pe^{-\gamma S_u}+(1-p)e^{-\gamma S_d}\right]$$

By sending $\gamma\to\infty$, it turns out

$$=\lim_{\gamma\to\infty}(S_0)=\lim_{\gamma\to\infty}\left\{-\frac{1}{\gamma(1+\gamma)}\right\}\left\{\ln\left[pe^{-\gamma[S_u-S_d]}+(1-p)\right]=\gamma\left[S_d\right]\right\}$$

After some calculation. We get

$$=\lim_{\gamma\to\infty}\left\{-\frac{1}{\gamma(1+\gamma)}\log\left[(1-p)+pe^{-\gamma[S_u-S_d]}\right]+\frac{S_d}{\gamma(1+\gamma)}\right\}$$

$$\lim_{\gamma\to\infty}(S_0)=\frac{S_d}{(1+\gamma)}. \tag{5.7}$$

So Formula (5.7) means: That if an agent is infinitely risk averse he or she should pay $\frac{|S_d|}{(1+\gamma)}$ for the asset to compensate.

Apply the second limit behavior of proposition 3.3 to the formula (5.6). which correpont to vanish the risk from the strategy, (sending $\gamma\to 0$).

Recall formula (5.6),

$$S_0=-\frac{1}{\gamma(1+r)}\log\left[pe^{-\gamma S_u}+(1-p)e^{-\gamma S_d}\right]$$

Take the limit with respect to $\gamma\to 0$,

$$\lim_{\gamma\to\infty}=\lim_{\gamma\to\infty}\left[-\frac{1}{\gamma(1+r)}\log\left[pe^{-\gamma S_u}+(1-p)e^{-\gamma S_d}\right]\right],$$

Applying Taylor expansion, this turns into

$$\lim_{\gamma\to\infty}(A_0) = -\frac{1}{\gamma(1+r)}\left\{\log\left(1-\gamma\left(pS_u+(1-p)S_d\right)\right)+o(\gamma)\right\}$$

Using the fact that $\log(1+y) = y + o(y)$, it gives

$$\begin{aligned}\lim_{\gamma\to\infty}(S_0) &= -\frac{1}{\gamma(1+r)}\log\left(1-\gamma\left(pS_u+(1-p)S_d\right)\right)\\ &= -\frac{1}{\gamma(1+r)}\gamma\left(pS_u+(1-p)S_d\right)\\ &= \frac{\left[pS_u+(1-p)S_d\right]}{(1+r)} = \frac{\mathbb{E}[S_1]}{(1+r)}\end{aligned} \quad . \tag{5.8}$$

Therefore, the formula (5.8) means that if we completely eliminate the risk formation strategy, the deterministic equivalent method becomes the risk neutral valuation method.

6 CONCLUSIONS

Through the analysis, we conclude that in the complete market, each claim can be perfectly replicated, and the whole risk can be eliminated from the strategy. When the market is incomplete, it is impossible to replicate completely, not all risks can be eliminated from the strategy, compared with the arbitrage pricing method and the indifference pricing method. The arbitrage pricing method is linear, but the undifferentiated pricing method is nonlinear. The indifference price is a nonlinear function of the claim's payoffs, exactly for $\propto \neq 0,1$, which is $\upsilon(\alpha C_T) \neq \alpha\upsilon(C_T)$. Indeed, as it is established by proposition 3.6, if $\propto > 1$, the indifference price is super-homogeneous, while, if $\propto > 1$, the indifference price is a sub-homogeneous, function of C_T. Another result, which was nonlinearity, for two payoffs, like C_T^1 and C_T^2 , the indifference price functional is nonadditive, namely $\upsilon\left(C_T^1+C_T^2\right) \neq \upsilon\left(C_T^1\right)+\upsilon\left(C_T^2\right)$, it can be extended to finite number of payoffs. The inadditive behavior of apathy pricing is a nonlinear characteristic of a direct consequence of apathy. In addition, as an important consequence of the nonlinearity, the agent does not want to double the risk if he wants to buy a claim. The undifferentiated price increases monotonously, but the monotonicity of the risk measure is reduced. Finally, the two limit behavior of the monotonicity of undifferentiated pricing is applied. Finally, a good application of the two restrictive behavior of the undifferentiated pricing rule can be modeled as an infinite risk aversion or a risk neutral.

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