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Mittag-Leffler Stability of Fractional Discrete Non-Autonomous Systems

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Abstract

In this paper, by using Lyapunov's direct method, we consider the Mittag-Leffler stability of fractional-order non-autonomous systems with the nabla left Caputo difference operators is studied. The sufficient conditions for Mittag-Leffler stability are discussed for such systems.

Keywords: Discrete Fractional Nabla Derivative; Fractional-Order Non-Autonomous Systems; Discrete Mittag-Leffler Function; Stability

INTRODUCTION

Many systems exhibit the fractional phenomena, such as motions in complex environments including random walk of bacteria in fractal substance, the viscoelastic materials, and electrode Celectrolyte polarization, and electromagnetic waves and so on. This makes the fractional order differential equations describe the dynamics of these systems more accurately than the integer order ones. Therefore, fractional calculus has become an important mathematical tool used in several branches to solve a variety of applied problems [1–5]. Starting from the idea of discretizing the Cauchy integral formula, Miller and Ross [6] obtained discrete versions of left type fractional sums and differences. The concept of Caputo fractional difference was introduced and investigated [7]. After that, Thabet investigated the relation between Riemann and Caputo fractional differences, and the delta and nabla discrete Mittag-Leffler functions are confirmed by solving Caputo type linear fractional difference equations [8].

Recently, the stability analysis of nonlinear fractional dynamic systems was introduced in many articles, including uniform stability, asymptotic stability, finite time stability and Mittag-Leffler stability and so on [9–15]. But the Mittag-Leffler stability of discrete fractional difference systems has not been studied.

This paper investigates the Mittag-Leffler stability of a class of fractional-order discrete nonautonomous systems with Caputo derivative. It is organized as follows. In section 2 basic definitions of discrete factional calculus are mentioned. In section 3, we discuss the Mittag-Leffler stability of the fractional discrete nonautonomous systems.

DEFINITIONS AND ESSENTIAL LEMMAS

In this paper, we use discrete fractional operator as our main tools, and summarize the basic theory of discrete calculus [16,17]. Throughout we assume that the functions are defined in the interval [a, b], and we denote that Na = {a, a + 1, . . .}.

For real number the a b, we denote that ¥ $_a$ = {a, a + 1, . . .} and $_{b-1}$¥ = {b, b −1, . . .},

For any real number the α rising function becomes

$$t^{\overline{\alpha}} = \frac{\Gamma(t+\alpha)}{\Gamma(t)}, t \in ¡ \ \backslash \{\ldots,-2,-1,0\}, \qquad 0^{\overline{\alpha}} = 0. \tag{1}$$

with the convention that division at pole yields zero. Given that the backward difference operator is defined by

$$\nabla f(t) = f(t) - f(t-1), \tag{2}$$

and then we observe the following:

$$\Delta(t^{\overline{\alpha}}) = \alpha t^{\overline{\alpha-1}}. \tag{3}$$

Definition 1 [16] Let $\sigma(t) = t+1$ and $\rho(t) = t-1$ be the forward and backward jumping operators. Then we have the following,

(i)The (nabla) left fractional sum of order $\alpha > 0$ (starting from a) is defined by

$$\nabla_a^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{t} (t-\rho(s))^{\overline{\alpha-1}} f(s), \qquad t \in \mathbb{N}_{a+1}. \tag{4}$$

(ii)The (nabla) right fractional sum of order $\alpha > 0$ (ending at b) is defined by

$${}_b\nabla^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=t}^{b-1} (s-\rho(t))^{\overline{\alpha-1}} f(s) = \frac{1}{\Gamma(\alpha)} \sum_{s=t}^{b-1} (\sigma(s)-t)^{\overline{\alpha-1}} f(s), \qquad t \in {}_{b-1}\mathbb{N}. \tag{5}$$

By the definition of the nabla right fractional sum, we observe the following:

(i) $\nabla_a^{-\alpha}$ maps functions defined on $\mathbb{N}_a$ to functions defined on $\mathbb{N}_a$.

(ii) ${}_b\nabla^{-\alpha}$ maps functions defined on ${}_b\mathbb{N}$ to functions defined on ${}_b\mathbb{N}$.

Definition 2 [16] The nabla α−order Riemann left fractional difference of a function f defined on $\mathbb{N}_a$ is defined by

$$\nabla_a^{\alpha} f(t) = \nabla^n \nabla_a^{-(n-\alpha)} f(t) = \frac{\nabla^n}{\Gamma(n-\alpha)} \sum_{s=a+1}^{t} (t-\rho(s))^{\overline{n-\alpha-1}} f(s), \qquad t \in \mathbb{N}_{a+1}. \tag{6}$$

By the definition of the nabla right fractional difference, we observe that ∇_a^{α} maps functions defined on $\mathbb{N}_a$ to functions defined on $\mathbb{N}_{a+n}$.

Definition 3 [17] The nabla $\alpha-$order Caputo left fractional difference of a function f defined on $\mathbb{N}_a$ and some points before a is defined by

$${}^C\nabla_a^{\alpha} f(t) = \nabla_a^{-(n-\alpha)} \nabla^n f(t) = \frac{1}{\Gamma(n-\alpha)} \sum_{s=a+1}^{t-(n-\alpha)} (t-\rho(s))^{\overline{n-\alpha-1}} \nabla^n f(s), \qquad t \in \mathbb{N}_{a+1}. \tag{7}$$

By the definition of the nabla Caputo left fractional difference, we observe that ∇_a^{α} maps functions defined on $\mathbb{N}_a$ to functions defined on $\mathbb{N}_{a+(n-\alpha)}$, where $n = [\alpha]+1$ and $[\alpha]$ is the greatest integer less than α.

Riemman and Caputo nabla left fractional differences are related by the following lemma.

Lemma 1 [17] For any $\alpha > 0$, the following identity is vaild:

$${}^C\nabla_a^{\alpha} f(t) = \nabla_a^{\alpha} f(t) - \sum_{k=0}^{n-1} \frac{(t-a)^{\overline{k-\alpha}}}{\Gamma(k-\alpha+1)} \nabla^k f(a). \tag{8}$$

In particular, when $0 < \alpha < 1$, one has

$${}^C\nabla_a^{\alpha} f(t) = \nabla_a^{\alpha} f(t) - \frac{(t-a)^{\overline{-\alpha}}}{\Gamma(1-\alpha)} f(a). \tag{9}$$

If $f(t) \geq 0$, we have ${}^C\nabla_a^{\alpha} f(t) \leq \nabla_a^{\alpha} f(t)$ for $t \geq t_0$.

Lemma 2 [17] Assume that $\alpha > 0$ and f is defined on suitable domains $\mathbb{N}_a$, the following identity is vaild:

$$\nabla_a^{-\alpha}\, {}^C\nabla_a^{\alpha} f(t) = f(t) - \sum_{k=0}^{n-1} \frac{(t-a)^{\overline{k}}}{k!} \nabla^k f(a). \tag{10}$$

In particular, when $0 < \alpha < 1$, one has

$$\nabla_a^{-\alpha}\, {}^C\nabla_a^{\alpha} f(t) = f(t) - f(a). \tag{11}$$

Proposition 1 [18] The generalized nabla discrete Mittag-Leffler function was defined as

$$E_{\overline{\alpha,\beta}}(\lambda, z) = \sum_{k=0}^{\infty} \lambda^k \frac{z^{\overline{k\alpha+\beta-1}}}{\Gamma(k\alpha+\beta)}. \tag{12}$$

For $\beta = 1$, one has

$$E_{\overline{\alpha}}(\lambda, z) = E_{\overline{\alpha,1}}(\lambda, z) = \sum_{k=0}^{\infty} \lambda^k \frac{z^{\overline{k\alpha}}}{\Gamma(k\alpha+1)}. \tag{13}$$

It should be mentioned that the solution of the following IVP(16)

$${}^C\nabla_a^{\alpha} x(t) = \lambda x(t) + f(t), \quad x(a) = x_0, \quad t \in \mathbb{N}_a. \tag{14}$$

Is given by

$$x(t) = x_0 E_{\overline{\alpha}}(\lambda, t-a) + \sum_{s=a+1}^{t} E_{\overline{\alpha,\alpha}}(\lambda, t-\rho(s)) f(s). \tag{15}$$

MITTAG-LEFFLER STABILITY FOR DISCRETE FRACTIONAL SYSTEMS

In this section, we consider the following nabla discrete fractional system

$${}^C\nabla_a^{\alpha} x(t) = f(t, x(t)), \quad x(a) = x_0, \quad t \in \mathbb{N}_a. \tag{16}$$

where $0 < \alpha < 1$, $f : \mathcal{T}_a \times \Omega \to \mathbb{R}^n$ is continuous, and $\Omega \in \mathbb{R}^n$ is a domain that contain the origin $x = 0$. Let $f(t, 0) = 0$, so that (16) admits the trivial solution. In order to analyze the stability of the fractional systems, we define the stability in the sense of Mittag-Leffler.

Definition 4 The trivial solution $x(t) = 0$ of fractional difference system (16) is said to be Mittag-Leffler stable if

$$\|x(t)\| \le \left[m(x(t_0)) E_{\overline{\alpha}}(\lambda, t-t_0) \right]^b, \tag{17}$$

Where $\lambda \ge 0, b > 0, m(0)=0, m(x) > 0, m(x)$ is locally Lipchitz on $x \in \Omega_r = \{x \in \mathbb{R}^n : \|x\| < r\} \subset \mathbb{R}^n$ and $\|\cdot\|$ denotes an arbitrary norm.

Theorem 1 Assume that there exists a scalar function $V(t, x) \in C[\mathcal{T}_a \times \Omega_r, \mathbb{R}_+]$ and positive constants c_1, c_2 and c_3 such that

$$c_1\|x\|^2 \le V(t,x) \le c_2\|x\|^2, \tag{18}$$

and

$$^{C}\nabla_a^{\alpha}V(t,x) \le c_3\|x\|^2, \tag{19}$$

for all $(t,x)\in ¥_a \times \Omega_r, t \ge t_0$, then the trivial solution of (16) is Mittag-Leffler stable.

Proof: Let $x(t) = x(t,t_0,x_0)$ be any solution of (16). From inequalities (18) and (19), we obtain

$$^{C}\nabla_a^{\alpha}V(t,x) \le \frac{c_3}{c_1}V(t,x). \tag{20}$$

Then there exists a nonnegative function $M(t)$ satisfying

$$^{C}\nabla_a^{\alpha}V(t,x) + M(t) = \frac{c_3}{c_1}V(t,x). \tag{21}$$

Taking the Laplace transform of (15) gives

$$V(t,x) = V(t_0,x_0)E_{\overline{\alpha}}(\frac{c_3}{c_1},t-a) - \sum_{s=a+1}^{t} E_{\overline{\alpha,\alpha}}(\frac{c_3}{c_1},t-a)M(s). \tag{22}$$

Obviously, $V(t,x) \le V(t_0,x_0)E_{\overline{\alpha}}(\frac{c_3}{c_1},t-a)$. From inequalities (18), we have $\|x\|^2 \le \frac{1}{c_1}V(t,x)$ and $V(t_0,x_0) \le c_2\|x_0\|^2$. And then

$$\|x\|^2 \le \frac{1}{c_1}V(t,x) \le \frac{1}{c_1}V(t_0,x_0)E_{\overline{\alpha}}(\frac{c_3}{c_1},t-a) \le \frac{c_2}{c_1}\|x_0\|^2 E_{\overline{\alpha}}(\frac{c_3}{c_1},t-a). \tag{23}$$

Therefore, that is $\|x(t)\| \le \sqrt{\frac{c_2}{c_1}}\|x_0\|\left[E_{\overline{\alpha}}(\frac{c_3}{c_1},t-a)\right]^{\frac{1}{2}}$. According to the definition 4, which implies that system (16) is Mittag-Leffler stable.

Theorem 2 Assume that there exists a scalar function $V(t,x) \in C\left[\mathcal{T}_a \times {}^n, ?_+\right]$ and positive constants c_1, c_2 and c_3 such that $c_1\|x\|^2 \le V(t,x) \le c_2\|x\|^2$ and $^{C}\nabla_a^{\alpha}V(t,x) \le c_3V(t,x)$, for all $(t,x) \in \mathcal{T}_a \times {}^n, t \ge t_0$, then the trivial solution of (16) is globally Mittag-Leffler stable.

Proof: Similar to the proof of Theorem 1, we can get

$$\|x(t)\| \le \sqrt{\frac{c_2}{c_1}}\|x_0\|\left[E_{\overline{\alpha}}(\frac{c_3}{c_1},t-a)\right]^{\frac{1}{2}}, \text{for all } (t,x) \in \mathcal{T}_a \times {}^n, t \ge t_0.$$

Thus $x(t)$ is globally Mittag-Leffler stable.

Theorem 3 (i) Assume that the assumptions in Theorem 1 are satisfied except replacing $^{C}\nabla_a^{\alpha}$ by ∇_a^{α}, then the trivial solution of (16) is Mittag-Leffler stable;

(ii) Assume that the assumptions in Theorem 2 are satisfied except replacing ${}^{C}\nabla_{a}^{\alpha}$ by ∇_{a}^{α}, then the trivial solution of (16) is globally Mittag-Leffler stable;

Proof: By using Lemma 1, the result is obviously valid. This completes the proof.

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REFERENCES

[1] Kilbas A A, Marichev O I, Samko S G. Fractional integral and derivatives (theory and applications) [J]. Gordon and Breach, Switzerland (1993).

[2] I Podlubny, Fractional Differential Equations, AP, New York, NY (1999).

[3] Srivastava H M, Trujillo J J. Theory and applications of fractional differential equations[J]. Elsevier Science (2006).

[4] Agrawal O P, Baleanu D. A Hamiltonian formulation and a direct numerical scheme for fractional optimal control problems[J]. Journal of Vibration and Control.13(9-10) (2000)1269-1281.

[5] Scalas E. Mixtures of compound Poisson processes as models of tick-by-tick financial data[J]. Chaos, Solitons and Fractals. 34(1) (2007)33-40.

[6] KS Miller and B Ross, Fractional Difference Calculus, Proceedings of the International Symposium on Univalent Functions, Fractional Calculus and their Applications, Nihon University, Koriyama, Japan (1988).

[7] Abdeljawad, Thabet. "On Riemann and Caputo fractional differences." Computers and Mathematics with Applications.62(3) (2011)1602-1611.

[8] Abdeljawad, Thabet. "On delta and nabla Caputo fractional differences and dual identities." Discrete Dynamics in Nature and Society (2013).

[9] Duarte-Mermoud, Manuel A., et al. "Using general quadratic Lyapunov functions to prove Lyapunov uniform stability for fractional order systems." Communications in Nonlinear Science and Numerical Simulation 22(1) (2015)650-659.

[10] Liu, Song, et al. "Lyapunov stability analysis of fractional nonlinear systems." Applied Mathematics Letters 51 (2016)13-19.

[11] Chen, Liping, et al. "Finite-time stability criteria for a class of fractional-order neural networks with delay." Neural Computing and Applications 27(3) (2016)549-556.

[12] Liu, Song, et al. "Stability of fractional nonlinear singular systems and its applications in synchronization of complex dynamical networks." Nonlinear Dynamics 84(4) (2016)2377-2385.

[13] Li Mengmeng, and JinRong Wang. "Finite time stability of fractional delay differential equations." Applied Mathematics Letters 64 (2017) 170-176.

[14] Jarad F, Abdeljawad T, Baleanu D. Stability of q-fractional non-autonomous systems[J]. Nonlinear Analysis: Real World Applications.14(1) (2013)780-784.

[15] Yang, Xujun, et al. "Mittag Leffler stability analysis of nonlinear fractional-order systems with impulses." Applied Mathematics and Computation 293 (2017) 416-422.

[16] T. Abdeljawad. On Riemann and Caputo fractional differences, computer and mathematics with applications. 2(2011) 1602-1611.

[17] Thabet Abdeljawad. On Delta and Nabla Caputo Fractional Differences and Dual Identities. Discrete Dynamics in Nature and Society. (2013) 1-12.

[18] M.Bohner, A. Peterson. Advances in Dynamic Equations on time scales, Birkhauser,Boston, Mass, USA, (2003).