

A Proof of Brouwer's Fixed Point Theorem Using Sperner's Lemma

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Abstract

This article offers a simple but rigorous proof of Brouwer's fixed point theorem using Sperner's Lemma. The general method I have used so far in the proof is mainly to convert the n -dimensional shapes to the corresponding case under the Sperner's Labeling and apply the Sperner's Lemma to solve the question.

Keywords: *Brouwer's Fixed Point Theorem; Sperner's Lemma; Proof*

INTRODUCTION

Brouwer's fixed point theorem, one of the earliest discoveries in algebraic topology, is first found and proved by the Dutch mathematician L.E.J. Brouwer in 1912. The history of it is closely interlinked with the 'nonlinear' behaviors of modern mathematics.

Inspired by the intermediate value theorem of Henri Poincaré (which states that if f is a continuous function whose domain contains the interval $[a, b]$, then it takes on any given value between $f(a)$ and $f(b)$ at some point within the interval) [1]. Brouwer investigated the characteristics of continuous functions and found that for any function f , as defined above there is always a point such that $f(x)=x$. This idea can also be expressed in an analogous way with more topological terminology, as every continuous function from a non-empty convex compact subset K of a Euclidean space to K itself has a fixed point.

The topic of invariant points in a function first came into scientist's views in 1880s, when the old problem of the stability of the solar system was attempted to find an exact solution. Poincaré, during that time, was investigating the motions of particles in a cup of coffee, which had some faint connections to the three-body-problem and found out that if the area is bounded, the path of a random point either appears stationary or converges to a limited cycle [2]. He then did further research and eventually obtained a result which is equivalent to Brouwer's theorem. Subsequent proofs came up at the start of twentieth century (three-dimensional case by Piers Bohr in 1904, the case of differentiable mappings of the n -dimensional closed ball was first proved in 1910 by Jacques Hadamard), and finally the generalized version for continuous mappings by Brouwer in 1911.

An illustrative approach to this theorem involving Sperner's Lemma will be depicted below.

1 PROOF

1.1 One-dimensional Case

Define a linear function $f(x)$, such that $x \in [a, b]$, $f(x) \in [a, b]$ and $f(x)$ is continuous,

Consider $g(x)=f(x)-x$.

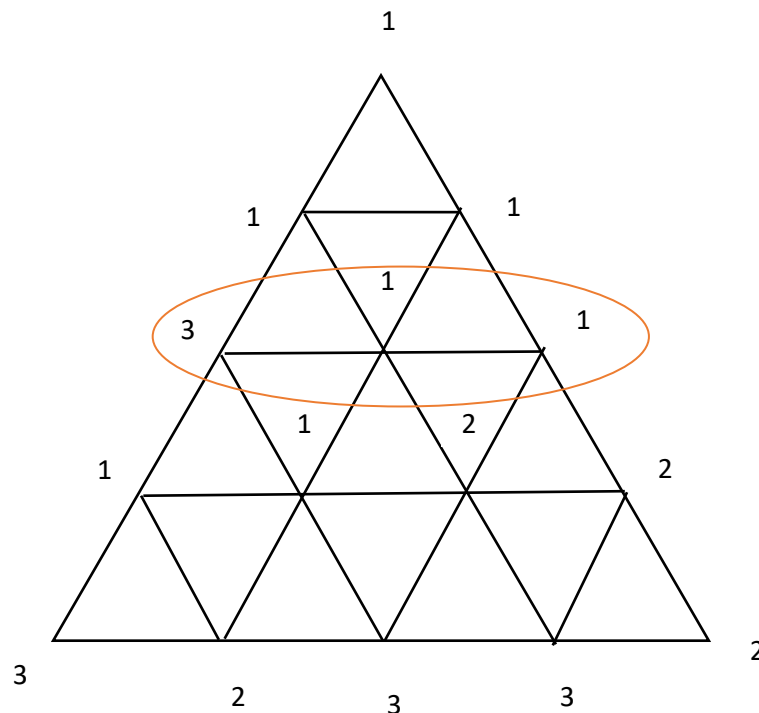
When $x=a$, then $g(x) \geq 0$, $x=b$, then $g(x) \leq 0$.

Conjecture 2.1. The Intermediate Value Theorem: If f a continuous function whose domain contains the interval $[a, b]$, then it can take on any given value between $f(a)$ and $f(b)$ at some point within the interval.

A deduction of this shows that if the sign of y changes inside an interval, given that the function is continuous, then there must be at least one intersection with the x -axis in that interval.

Therefore, there exists a value of x such that $g(x)=0$, indicating $f(x)=x$, which means that there is always an invariant point in any one-dimensional function.

1.2 Two-dimensional Case



Construct a triangle, assign every vertex and every intersection inside it with a number from 1 to 3. The numbers of every point can only be chosen from the values of the two vertices of the line on which the point is located on. For instance, if we look at the second row on the graph, the numbers on the two ends are 1 and 3, so the number assigned to the point in the middle can be either 1 or 3. After that, we look at the colored lines whose intersection is the point in the middle. The number in common on their vertices is 1, so the middle point must be 1. This method of assigning is called Sperner Labeling [3].

We call the shortest line segment which contains different numbers on the vertices of the original line a ‘sub-line’ or ‘sub-interval’, and the new triangle created which has different numbers in all three vertices ‘sub-triangles’. There is always a sub-triangle in the original one no matter how the triangulations are drawn.

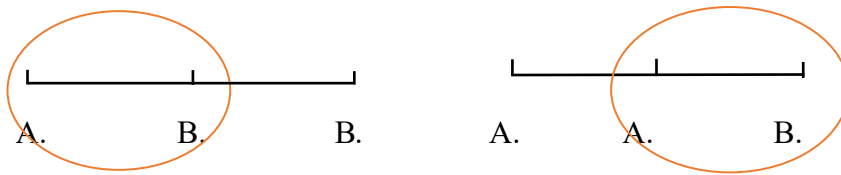
Proposition 2.1. If a line in this labeled triangle has been assigned two different numbers, it must have an odd number of sub-intervals.

Proof. This statement can be easily demonstrated by induction.

Assume line AB always has an odd number of sub-intervals under Sperner Labeling. (A and B are some random rational numbers)

Definition: An equinox point: A point which divides a line segment into two parts, which are not necessarily equal.

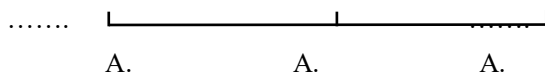
(i) $n=1$ (there is one equinox point on the line), no matter whether A or B is assigned to that point, there is always one sub-line in the line. 1 is odd.



(ii) $n=k$, there is k equinox points on the line and an odd number of sub-intervals.

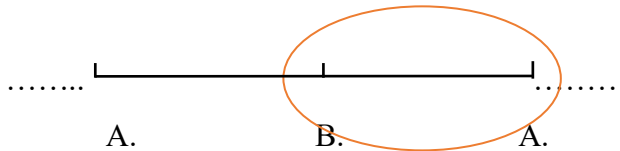
(iii) $n=k+1$, might have 3 cases.

a. The equinox point is added to the middle of two points with identical numbers. All three of them have the same number A.



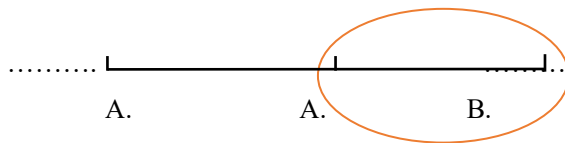
The number of sub-intervals remains the same. (odd)

b. The equinox point, which has a different number is added to the middle of two points with identical numbers.



The number of sub-intervals increases by 2. (odd)

c. The equinox point is added to the middle of two points with different numbers.



The number of sub-intervals remains unchanged. (odd)

In conclusion, if this statement works for $n=k$, then it works for $n=k+1$.

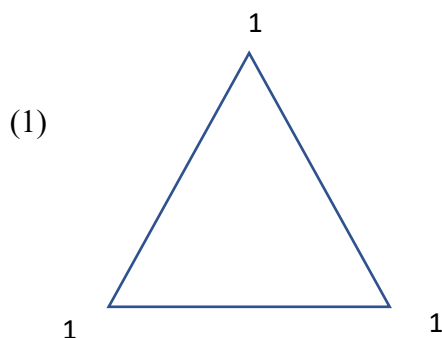
\therefore It works for $n=1$

\therefore It works for all positive integer n . If a line in this labeled triangle has been assigned two different numbers, it must have an odd number of sub-intervals.

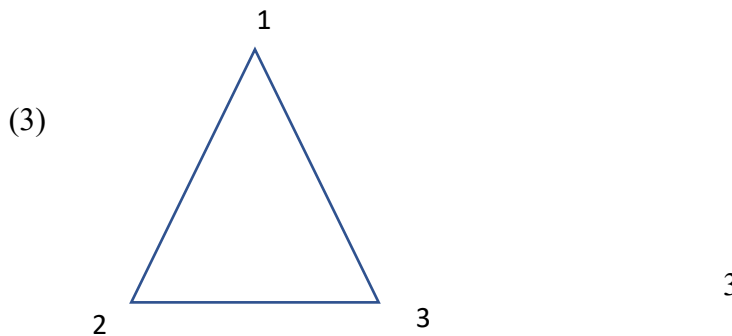
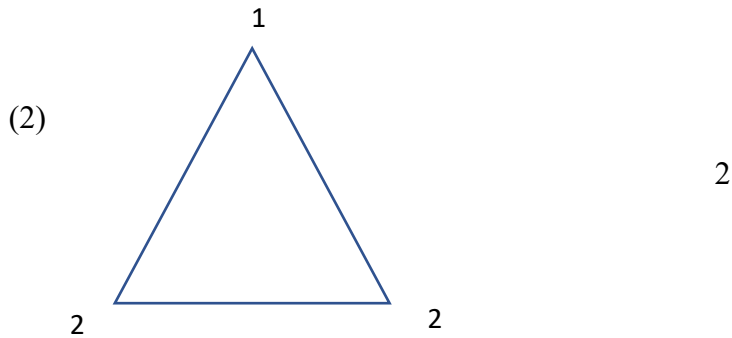
Proposition 2.2. There is always a sub-triangle in the original one no matter how the triangulations are drawn.

Proof. 3 kinds of triangles can be created.

Number of lines containing different numbers on the vertices (sub-line)



0



A proof by contradiction is used to solve the problem.

Assume that there is no sub-triangle in any triangle under Sperner's labeling, which can be paraphrased as 'all the triangles created are either (1) or (2). The total number of sub-lines = $2 \times$ inner sides (because they belong to 2 triangles) + outer sides = even (the sum of any multiple of 0 and/or 2 must be even.)

However, according to what we have already proved earlier, the total number of sub-lines must be odd for a line with different numbers on its vertices.

\therefore Contradiction. There must be a sub-triangle with different numbers on its vertices. Thus, after undergoing a series of divisions and transformations, there always exists a new triangle with 3 different numbers. In other words, $f(x) = x$ in two dimensions.

1.3 Three-dimensional Case

Define a function such that $x \in D^2$, $f(x) \in D^2$ (Both x and $f(x)$ belong to the unit circle.) Construct a triangle with vertices $(1,0,0)$ $(0,1,0)$ and $(0,0,1)$. $\Delta = \{(x, y, z) | x+y+z=1, 0 \leq x, y, z \leq 1\}$

As the unit circle and this triangle are homeomorphic, they can be viewed as isomers in the category of topological spaces, which means they are mappings with all the characteristics of the given space. Therefore, the theorem in three dimensions can be proved using this triangle as well. Assume that $f(\Delta) = \Delta$.

Just like in the previous case, use the proof of contradiction. Assume that for all $x \in \Delta$, $f(x) \neq x$.

Define the component of a certain point $x (x_1, x_2, x_3)$, $x_1 + x_2 + x_3 = 1 (0 \leq x_i \leq 1)$

$f(x) (y_1, y_2, y_3)$, $y_1 + y_2 + y_3 = 1 (0 \leq y_i \leq 1)$

There must be at least one $y_i \leq x_i$. Assign the minimum value of i to this point. For the three vertices, the values are:

A $(1,0,0)$ B $(0,1,0)$ C $(0,0,1)$

$x=1$ $x=2$ $x=3$

Therefore, all the points on AB possess the format $(a, b, 0) - x \neq 3$.

All the points on AC possess the format $(a, 0, b) - x \neq 2$.

All the points on BC possess the format $(0, a, b) - x \neq 1$.

The whole process mentioned above constructs a Sperner Triangle. Every point has been assigned a number which is equal to either value of the number on the vertices of the line segment to which this point belongs, which turns the question from three-dimensional to two dimensional, and as we have already proved that there is always a sub-triangle in a triangle with Sperner's Labeling, the result also works for this.

1.4 Generalized Form

As in this context, the whole proof is on the basis of Sperner's Lemma, we might need to prove the multidimensional Sperner's Lemma as well. First of all, we create a shape of n-dimension and assign every vertex to a different number. The number of every point on one single line is only chosen from the number of the two ends of that line. Now a Sperner Labeling is constructed. Due to what has been proved earlier in the two-dimensional case, the sum of sub-intervals must be odd. Once again, a proof of contradiction is applied to solve this.

First of all, assume that there is no triangulation which has different numbers assigned to all three vertices. As is mentioned above when we were proving the two-dimensional case that there are only two cases which meet the condition. There are 0 or 2 lines with different numbers on two ends in each triangulation. Therefore, the sum of all these lines must be an even number. However, we have also proved that if every line has two different numbers assigned to both ends, the sum of the lines (ie. sub-intervals) has to be odd.

∴ Contradiction. Every n-dimensional simplex has one or more (odd number) triangulations which has different numbers assigned to every vertex. Hence, every n-dimensional simplex has at least one sub-simplex which has different numbers on vertices.

Now it's time for us to come back to our original n- dimensional proof for Brouwer's theorem. To begin with, construct a simplex Δ^n in n-dimension. The points in this simplex are defined as $\Delta^n = \{(x_1, x_2, x_3, \dots, x_n) \mid x_1 + x_2 + x_3 + \dots + x_n = 1, 0 \leq x_1, x_2, x_3, \dots, x_n \leq 1\}$ x_i are the component vectors of every point. Correspondingly, the values of $f(x)$ after some manipulations are defined as $\{(y_1, y_2, y_3, \dots, y_n) \mid y_1 + y_2 + y_3 + \dots + y_n = 1, 0 \leq y_1, y_2, y_3, \dots, y_n \leq 1\}$. Thus, by the Pigeonhole principle, for every point, there is always at least one value of y_i such that $y_i \leq x_i$. Assign the minimum value of i which meets the requirement to the point. By doing so, a simplex with Sperner's Labeling is constructed, and according to the proof of Sperner's Lemma for multidimensional spaces, there is always a triangulation which has different numbers assigned to all three vertices.

The proof is therefore complete.

2 CONCLUSION

The general method I have used so far in the proof is mainly to convert the n-dimensional shapes to the corresponding case under the Sperner's Labeling and apply the Sperner's Lemma to solve the question.

Furthermore, there are lots of equivalents to the theorem, such as Knaster-Kuratowski-mazurkiewicz lemma [4], which is proposed by three mathematicians Knaster, Kuratowski and Mazurkiewicz and is about set covering, Borsuk-Ulam theorem which involves algebraic topology, and Tucker's Lemma, which is based on combinatorics. Most of which can be proved or used to prove Brouwer's fixed point theorem, indicating that the maths in this field is interlinked. Topology can be applied to lots of areas in real life, like working on the sequences of DNA in biology, and studying

the complicated relationships among different regions in an area in geographic information system (GIS) [5]. This branch of mathematics will contribute hugely to the development of human society in the future and make magnificent progress on scientific research as well.

ACKNOWLEDGEMENT

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